MATHEMATICS 3103 (Functional Analysis)
YEAR 2012–2013, TERM 2

PROBLEM SET #1

This problem set is due at the *beginning* of class on Monday 21 January. I urge you to start on it *early* in the week, as some of the problems are not entirely trivial. Only Problems 3, 4, 5 and 7 will be formally assessed, but I strongly urge you not to neglect the others!

Topics:

- Review of set theory: equivalence of sets (i.e. having the same cardinality); countably infinite and uncountably infinite sets.
- What is functional analysis?
- Metric spaces: elementary properties (review), separability, completeness.

Readings:

- Handout #0: Review of set theory.
- Kolmogorov–Fomin, Sections 1.1 and 1.2 (handout).
- Vilenkin, *Stories about Sets*, Chapters 1–3 (handout). This book gives an entertaining introduction to the theory of infinite sets. But of course all the mathematical material is contained in Kolmogorov–Fomin.
- Handout #1: What is Functional Analysis?
- I strongly urge you to consult the chapter on metric spaces in one of the suggested textbooks, e.g. Kolmogorov–Fomin, Chapter 2; Kreyszig, Chapter 1; or Dieudonné, Chapter 3.

1. (a) Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be finite sequences of real numbers. Prove that

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 = \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i b_j - b_i a_j)^2 .$$

Then deduce the Cauchy–Schwarz inequality from this identity.

(b) Let f and g be continuous real-valued functions on the interval $[a,b]$ of the real line. Prove that

$$\left(\int_a^b f(t)g(t)\,dt\right)^2 = \left(\int_a^b f(t)^2\,dt\right)\left(\int_a^b g(t)^2\,dt\right) - \frac{1}{2}\int_a^b\int_a^b [f(s)g(t)-g(s)f(t)]^2\,ds\,dt\ .$$

Then deduce the Cauchy–Schwarz inequality for integrals from this identity.

2. (a) Let (X_1,d_1), $\ldots$, (X_n,d_n) be metric spaces, and let X be the Cartesian-product space $X_1 \times \cdots \times X_n$ [that is, the space consisting of n-tuples $x = (x_1,\ldots,x_n)$ with $x_i \in X_i$]. Show that

$$\mathbf{d}_1(x,y) = \sum_{i=1}^n d_i(x_i,y_i)$$

$$\mathbf{d}_\infty(x,y) = \max_{1\le i\le n} d_i(x_i,y_i)$$

are metrics on X, and that

$$\mathbf{d}_\infty(x,y) \le \mathbf{d}_1(x,y) \le n\,\mathbf{d}_\infty(x,y)\ .$$

(b) Let (X,d) be an arbitrary metric space, and define

$$d'(x,y) = \frac{d(x,y)}{1+d(x,y)}\ .$$

Show that d' is a metric on X, and that it is **equivalent** to d in the sense that (X,d) and (X,d') have exactly the same open sets. [Note that d' is **bounded** since we always have $0 \le d'(x,y) < 1$.]

(c) Let (X_1,d_1), (X_2,d_2), $\ldots$ be an infinite sequence of metric spaces, and let X be the Cartesian-product space $X_1 \times X_2 \times \cdots$ [that is, the space consisting of infinite sequences $x = (x_1,x_2,\ldots)$ with $x_i \in X_i$]. Show that

$$d(x,y) = \sum_{j=1}^\infty \frac{1}{2^j}\,\frac{d_j(x_j,y_j)}{1+d_j(x_j,y_j)}$$

is a metric on X. [In particular, this completes the proof that our distance function on the space $\mathbb{R}^{\mathbb{N}}$ (Example 15) is indeed a metric.]

3. (a) Find disjoint closed sets $A,B \subseteq \mathbb{R}$ such that $d(A,B) = 0$.

(b) What if it is required, in addition, that at least one of the sets A,B be *bounded*? Can you do it now?

4. Let A be a set in a metric space X, and let $x \in \overline{A} \setminus A$ (i.e. x is a cluster point of A but does not belong to A). Prove that every neighborhood of x contains infinitely many points of A.

5. Prove that the space $C(\mathbb{R})$ of bounded continuous real-valued functions on the real line $\mathbb{R}$, equipped with the sup metric, is not separable. [*Hint:* Imitate the proof given in class for ℓ^∞.]

6. If A is a subset of a metric space X, a point $x \in A$ is said to be an **isolated point** of A if there exists a neighborhood V of x in X such that $V \cap A = \{x\}$. Prove that if X is a separable metric space, and A is a subset of X with the property that *all* its points are isolated points, then A is countable. [*Hint:* Imitate the construction used in the proof that ℓ^∞ is nonseparable, i.e. construct a family of disjoint open balls centered at the points of A. What should the radii of those balls be? For $x \in A$, define $r(x) = d(x, A \setminus \{x\})$ and observe that $r(x) > 0$ (why?). Balls of radii $r(x)$ are not in general disjoint; but perhaps, inspired by the ℓ^∞ proof, you can find suitable smaller radii such that the balls are disjoint.]

7. (a) Prove that the sequence space ℓ^1 is separable. [*Hint:* Imitate the proof given in class for c_0.]

   (b) Do the same for ℓ^2.

8. Let X be a metric space. A collection $\mathcal{G}$ of nonempty open sets of X is called a **basis** for the open sets of X if every nonempty open set of X is the union of some subcollection of the collection $\mathcal{G}$.

   (a) Prove that a collection $\mathcal{G}$ of nonempty open sets is a basis if and only if, for every $x \in X$ and every neighborhood V of x, there exists $U \in \mathcal{G}$ such that $x \in U \subseteq V$.

   (b) Prove that X is separable if and only if there exists a countable basis for the open sets of X.

   (c) Prove that any subspace of a separable metric space is separable. [*Hint:* Use (b) and Lemma 1.16.]

   Remark: A topological space is called **separable** if it has a countable dense set, and is called **second countable** if it has a countable basis for its open sets. So what you have shown in (b) is that, *for metric spaces*, separablity and second countability are equivalent properties. For general topological spaces, by contrast, this is *not* true: second countability implies separability (by the same argument you used here), but the converse is *not* in general true.