All questions may be attempted, but only marks obtained on the best **four** *solutions will count.*

The use of an electronic calculator is **not** *permitted in this examination.*

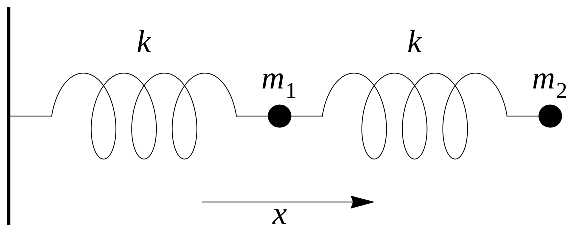
1. In this problem $\mathbf{r}$ denotes the position of a particle, $r = |\mathbf{r}|$ its magnitude, and $\widehat{\mathbf{e}}_r = \mathbf{r}/r$ the corresponding unit vector.

   (a) Show that
   $$\frac{d}{dt}\widehat{\mathbf{e}}_r \;=\; \frac{1}{r}\frac{d\mathbf{r}}{dt} \;-\; \frac{1}{r^2}\frac{dr}{dt}\,\mathbf{r}\;.$$

   (b) Suppose that a particle of mass m moves under a central force $\mathbf{F} = F(r)\widehat{\mathbf{e}}_r$. Show that
   $$\frac{d}{dt}(\mathbf{p}\times\mathbf{L}) \;=\; -mr^2F(r)\,\frac{d}{dt}\,\widehat{\mathbf{e}}_r$$
   where $\mathbf{p}$ is the particle's momentum and $\mathbf{L}$ is its angular momentum. [You may find the identity $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$ useful.]

   (c) If the force law is $F(r) = -k/r^2$, what can be deduced about the time evolution of the vector $\mathbf{A} = \mathbf{p}\times\mathbf{L} - mk\widehat{\mathbf{e}}_r$?

2. Two particles, of masses m_1 and m_2, respectively, are connected to a fixed wall by springs of spring constant k, as shown in the diagram:



   The particles move horizontally. Let x_1 and x_2 be the displacements of the two particles from their equilibrium positions.

   (a) Derive the equations of motion. (You may use either Newtonian or Lagrangian methods.)

   (b) Find the frequencies of the normal modes.

   (c) In the special case $m_1/m_2 = 3/2$, find the eigenvectors corresponding to the normal modes.

3. A particle of mass m is hung from the ceiling via an inextensible massless string of length ℓ, and it swings in a vertical plane subject to the Earth's gravity. Let θ be the angle made by the string relative to the vertical, so that the equation of motion is

$$\ddot{\theta} + \frac{g}{\ell} \sin\theta \;=\; 0 \;.$$

We are interested in the oscillatory motion of amplitude A (i.e., back and forth between angles $\theta = \pm A$).

(a) Find an expression for the period of oscillation T as a definite integral. (You need not attempt to evaluate this integral!)

(b) Use perturbation theory to find the motion $\theta(t)$ with initial conditions $\theta(0) = A$, $\dot{\theta}(0) = 0$ through order A^3, where A is considered "small". [*Hint:* $\cos^3\psi = \frac{1}{4}\cos 3\psi + \frac{3}{4}\cos\psi$.] Explain what a "secular term" is, and say which term in your answer is the secular term.

(c) Use the Lindstedt renormalization procedure to compute the frequency of oscillation ω through order A^2.

4. A wire of shape $z = f(x)$ rotates with angular frequency ω about a vertical axis passing through $x = 0$. A bead of mass m slides frictionlessly on the wire under the influence of gravity. Let ρ be the bead's horizontal distance from the vertical axis.

(a) Using ρ as the generalised coordinate, find the kinetic energy, the potential energy, and the Lagrangian.

(b) Show that the equation of motion is

$$[1 + f'(\rho)^2]\ddot{\rho} \;+\; f'(\rho)f''(\rho)\dot{\rho}^2 \;-\; \omega^2\rho \;+\; gf'(\rho) \;=\; 0 \;.$$

(c) Fix some value $\rho_0 > 0$. What value of ω will allow the bead to remain at $\rho = \rho_0$?

(d) If ω is chosen as in part (c), under what conditions is the solution $\rho = \rho_0$ a stable equilibrium? When it is stable, find the frequency of small oscillations about the solution $\rho = \rho_0$.

[Note that you may be able to do parts (c) and (d) even if you were unable to do (a) and (b).]

5. A particle of mass m moves on a smooth horizontal table. It is connected to a massless inextensible string that passes through a small hole in the table, and the string is pulled from below in such a way that the particle's distance from the hole is a specified function $R(t)$. Use polar coordinates (r, θ) with the origin located at the hole.

   (a) Using θ as the generalised coordinate, find the kinetic energy, the potential energy, and the Lagrangian.

   (b) Show that θ is a cyclic coordinate, and find the corresponding conserved momentum p_θ. What is the physical meaning of p_θ?

   (c) Find the Hamiltonian and the Hamilton equations of motion.

   (d) Compare the Hamiltonian and the total energy. Is the Hamiltonian conserved? Is the total energy conserved? Justify your answers, and explain physically.

6. Let $\mathbf{q} = (q_1, \ldots, q_n)$ and $\mathbf{p} = (p_1, \ldots, p_n)$ be canonical coordinates.

   (a) Define the Poisson bracket $\{f, g\}$ of a pair of functions $f(\mathbf{q}, \mathbf{p})$ and $g(\mathbf{q}, \mathbf{p})$.

   (b) Using Hamilton's equations, show that $\dfrac{df}{dt} = \{f, H\}$ where H is the Hamiltonian.

   (c) Show that $\{fg, h\} = f\{g, h\} + \{f, h\}g$ for any three functions $f(\mathbf{q}, \mathbf{p})$, $g(\mathbf{q}, \mathbf{p})$, $h(\mathbf{q}, \mathbf{p})$.

   For the remainder of this problem, suppose that $\mathbf{q} = (q_1, q_2, q_3)$ are Cartesian coordinates for a single particle, and that $\mathbf{p} = (p_1, p_2, p_3)$ is the particle's momentum.

   (d) Express the angular momentum $\mathbf{L}$ in terms of $\mathbf{q}$ and $\mathbf{p}$, and compute the Poisson brackets $\{q_i, L_j\}$ and $\{p_i, L_j\}$. [You may wish to express your answers in terms of the antisymmetric symbol ϵ_{ijk}.]

   (e) Show that $\{L_i, L_j\} = \epsilon_{ijk} L_k$.

   (f) Show that $\{L_i, |\mathbf{L}|^2\} = 0$. [*Hint*: The identity $\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$ may be useful.]