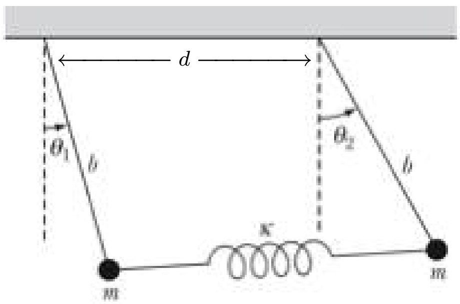
All questions may be attempted, but only marks obtained on the best **four** *solutions will count.*

The use of an electronic calculator is **not** *permitted in this examination.*

1. Two simple pendula of length b, each with a bob of mass m, are hung from a rigid horizontal rod a distance d apart. The two bobs are connected by an ideal spring whose equlibrium length is d and whose spring constant is κ. The two pendula swing in a vertical plane containing the rod. Let θ_1 and θ_2 be the angles of the two pendula relative to the vertical.



(a) Using θ_1 and θ_2 as the generalised coordinates, compute the kinetic energy and the potential energy.

(b) Expand the potential energy through quadratic order in θ_1 and θ_2, to show that

$$U \;=\; -2mgb \;+\; \frac{1}{2}mgb(\theta_1^2+\theta_2^2) \;+\; \frac{1}{2}\kappa b^2(\theta_2-\theta_1)^2 \;+\; O(\theta^3)$$

(c) Show, using either Newtonian or Lagrangian methods, that the equations of motion in the linearised approximation are

$$\begin{aligned} mb^2\ddot{\theta}_1 &= -mgb\theta_1 \;-\; \kappa b^2(\theta_1-\theta_2) \\ mb^2\ddot{\theta}_2 &= -mgb\theta_2 \;+\; \kappa b^2(\theta_1-\theta_2) \end{aligned}$$

[Note that you may be able to do this part even if you were unable to do (b).]

(d) In the linearised approximation, find the normal modes and the associated frequencies of oscillation. [Note that you may be able to do this part even if you were unable to do (a)–(c).]

2. A chandelier of mass m is attached to the end of a rope that passes through a fixed pulley on the ceiling. A person pulls on the other end of the rope at a constant rate α, so that the chandelier's distance from the pulley is $\ell(t) = \ell_0 - \alpha t$. The chandelier swings in a fixed plane; let θ denote the chandelier's angle relative to the vertical. Assume a uniform gravitational field g, pointing downwards.

   (a) Use Newtonian methods to find the equation of motion for θ.

   (b) Find the Lagrangian and the Lagrange equations of motion. Compare to part (a).

   (c) Find the Hamiltonian and the Hamilton equations of motion. Compare to parts (a) and (b).

   (d) Compare the Hamiltonian and the total energy. Is the Hamiltonian conserved? Is the total energy conserved? Justify your answers.

3. A particle of mass m moves along a line subject to the potential energy

$$U(x) \;=\; \frac{1}{2}kx^2 \;+\; \frac{\lambda}{4}x^4$$

   where $\lambda > 0$. We are interested in the oscillatory motion of amplitude A.

   (a) Find an expression for the period of oscillation T as a definite integral. (You need not attempt to evaluate this integral!)

   (b) Use perturbation theory to find the motion $x(t)$ with initial conditions $x(0) = A$, $\dot{x}(0) = 0$ through first order in λ, where λ is considered "small". [*Hint:* $\cos^3\theta = \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta$.] Explain what a "secular term" is, and say which term in your answer is the secular term.

   (c) Use the Lindstedt renormalization procedure to compute the frequency of oscillation ω through first order in λ.

4. A circular hoop of radius R rotates with angular frequency ω about a vertical axis coincident with its diameter. A bead of mass m slides frictionlessly under gravity on the hoop. Let θ be the bead's angular position relative to the vertical (so that $\theta = 0$ corresponds to the bead being at the bottom of the hoop).

   (a) Using θ as the generalised coordinate, find the kinetic energy, the potential energy, and the Lagrangian.

   (b) Show that the equation of motion is

$$mR^2\ddot{\theta} + mgR\sin\theta - m\omega^2R^2\sin\theta\cos\theta = 0 .$$

   (c) Under what conditions can the bead sit at a fixed angle θ_0 not equal to 0 or π? Express θ_0 as a function of ω.

   (d) In the situation of part (c), determine the frequency of small oscillations about θ_0.

   [Note that you may be able to do parts (c) and (d) even if you were unable to do (a) and (b).]

5. Let $\mathbf{q} = (q_1, \ldots, q_n)$ and $\mathbf{p} = (p_1, \ldots, p_n)$ be canonical coordinates.

   (a) Define the Poisson bracket $\{f, g\}$ of a pair of functions $f(\mathbf{q}, \mathbf{p})$ and $g(\mathbf{q}, \mathbf{p})$.

   (b) Using Hamilton's equations, show that $\dfrac{df}{dt} = \{f, H\}$ where H is the Hamiltonian.

   (c) Show that $\{fg, h\} = f\{g, h\} + \{f, h\}g$ for any three functions $f(\mathbf{q}, \mathbf{p})$, $g(\mathbf{q}, \mathbf{p})$, $h(\mathbf{q}, \mathbf{p})$.

   For the remainder of this problem, suppose that $\mathbf{q} = (q_1, q_2, q_3)$ are Cartesian coordinates for a single particle, and that $\mathbf{p} = (p_1, p_2, p_3)$ is the particle's momentum.

   (d) Express the angular momentum $\mathbf{L}$ in terms of $\mathbf{q}$ and $\mathbf{p}$, and compute the Poisson brackets $\{q_i, L_j\}$ and $\{p_i, L_j\}$. [You may wish to express your answers in terms of the antisymmetric symbol ϵ_{ijk}.]

   (e) Show that $\{L_i, L_j\} = \epsilon_{ijk} L_k$.

   (f) Show that $\{L_i, |\mathbf{L}|^2\} = 0$. [*Hint*: The identity $\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$ may be useful.]

6. A symmetrical top, with principal moments of inertia $I_1 = I_2$ and I_3, is supported at a fixed point P; the motion takes place in a uniform gravitational field g. In terms of the Euler angles θ, φ and ψ, the Lagrangian is

$$L \;=\; \tfrac{1}{2}I_1(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta) \;+\; \tfrac{1}{2}I_3(\dot{\psi} + \dot{\varphi}\cos\theta)^2 \;-\; mga\cos\theta$$

where a is the distance from the top's centre of mass to the support point P.

(a) Find the three generalised momenta p_θ, p_φ and p_ψ.

(b) Use Lagrange's equations to show that p_φ and p_ψ are constants of motion.

(c) Find the Hamiltonian $H(\theta, \varphi, \psi, p_\theta, p_\varphi, p_\psi)$, and use Hamilton's equations to show that p_φ and p_ψ are constants of motion.

(d) Deduce that the Hamiltonian can be written in the form

$$H \;=\; \frac{p_\theta^2}{2I_1} \;+\; U(\theta)\ ,$$

and find the "effective potential" $U(\theta)$ [in which p_φ and p_ψ will appear as parameters].

(e) Use Hamilton's equations to show that

$$I_1\ddot{\theta} \;=\; -U'(\theta)\ .$$

(f) The top is put into motion with initial conditions $\theta(0) = \theta_0$ and $\dot{\theta}(0) = 0$, where θ_0 is a local minimum of $U(\theta)$. Describe the subsequent motion of the top.