Math 1302, Week 9: A brief introduction to waves

There are two main types of waves that we'll consider:

Travelling wave such as a ripple on the surface of water, or a Mexican wave. Such waves propagate in space with time, but without changing their profile;

Stationary/Standing wave Such as the wave formed by plucking a rubber band fixed between two points. This is also the kind of wave that forms in an organ pipe. These waves do not propagate, but appear to be stationary.

Travelling waves

A Mexican wave is a useful guiding example. Consider a stadium of people. The wave is formed by successive neighbours standing up and then sitting down. Thus each person performs the same action, but at a slightly later time than their neighbour. Note also that they remain in their seat location - they do not move laterally. This is the same for a water wave: water particles do not move horizontally; they simply move up and down at different time instants. As an example, we consider the function $y(x,t) = Ae^{-(x-ct)^2}$ where $c > 0$. Figure 1 shows the

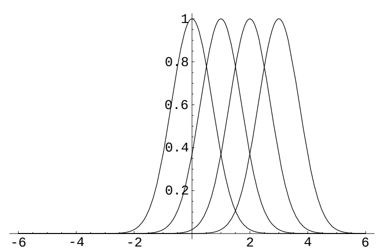

Figure 1: Travelling wave moving to the right with unit speed, but maintaining its profile as it moves. The profile here is $f(x) = e^{-x^2}$.

function $y(x,t)$ as a function of x for $t = 0, 1, 2, 3$. As is clear from the figure, $y(x,t)$ is simply the curve $y(x,0)$ translated by ct units to the right. Thus $y(x,t)$ represents a travelling wave with profile $f(x) = y(x,0)$ that moves to the right with speed c. When $c < 0$ the wave simply moves to the left with speed $-c(> 0)$. In fact for any function $f(x)$ we may construct a travelling wave $y(x,t) = f(x - ct)$.

Standing waves

For example, when we stretch a rubber band between two points and pluck it we get a wave that does not appear to propagate. Instead, if we consider different "particles" along the band, a given particle oscillates up and down repetitively (unlike with a single water ripple where a water particle oscillates once and then rests). However, the amplitude of the rubber band particle depends on its position along the band (whereas for a travelling wave, every particle under goes the same amplitude of oscillation regardless of its position). This is an example of a standing wave. The displacement y of a standing wave has the form $y(x,t) = F(x)G(t)$ for some functions F, G with G periodic in t. Thus for fixed x, $y(x,t)$ is periodic. A common example is $y(x,t) = A\cos kx \cos \omega t$.

The Wave Equation: Waves on an elastic string

We will derive a differential equation for the mechanics of a string undergoing low amplitude transverse vibrations (in the absence of gravity). The equation will describe the vertical height $y(x,t)$ of the string at point x along the string at a given time t. Since y depends on both x and t we will have a partial differential equation. We refer to figure 2. Suppose that the

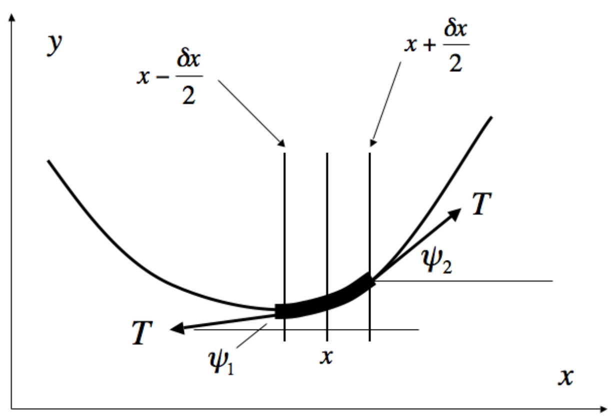

Figure 2: Small amplitude waves on a string (not to scale.)

string has constant mass ρ per unit length. Consider a small section of the string of length $\delta s \approx \delta x$ (since the string is very near to horizontal) and the forces acting on it. It has mass $\rho\delta x$ and vertical acceleration $\dfrac{\partial^2 y}{\partial t^2}$ (this is a second partial derivative and not $\ddot{y}$ because now y is a function of two variables x and t. By $\partial^2 y(x,t)/\partial t^2$ we simply mean second derivative at t treating x as constant). We assume that the displacement of the string is small and so that the tension remains a uniform T along the string. The angles ψ_1, ψ_2 shown are small for small transverse displacements and are exaggerated in the figure.

By Newton's 2nd law, to first order, the vertical mass times acceleration is

$$\rho\delta x \frac{\partial^2 y}{\partial t^2} = T\sin(\psi_2) - T\sin(\psi_1).$$

Since ψ_1, ψ_2 are small we have to first order ($\sin\psi_1 \approx \psi_1$ and $\sin\psi_2 \approx \psi_2$)

$$\rho\delta x \frac{\partial^2 y}{\partial t^2} = T(\psi_2 - \psi_1). \tag{1}$$

On the other hand,

$$\begin{aligned}\frac{\partial y}{\partial x}(x + \delta x/2, t) &= \tan(\psi_2) = \frac{\sin(\psi_2)}{\cos(\psi_2)} \approx \frac{\psi_2}{1} = \psi_2 \\ \frac{\partial y}{\partial x}(x - \delta x/2, t) &= \tan(\psi_1) = \frac{\sin(\psi_1)}{\cos(\psi_1)} \approx \frac{\psi_1}{1} = \psi_1.\end{aligned}$$

Now[1]

$$\frac{\partial y}{\partial x}(x + \delta x/2, t) - \frac{\partial y}{\partial x}(x - \delta x/2, t) = \frac{\partial^2 y}{\partial x^2}(x, t)\delta x + O(\delta x^2), \tag{2}$$

Hence from (1) and (2) we obtain

$$\rho\delta x \frac{\partial^2 y}{\partial t^2} = T\frac{\partial^2 y}{\partial x^2}\delta x + O(\delta x^2),$$

so that gathering together 1st order terms in δx and then letting $\delta x \to 0$

$$\rho\frac{\partial^2 y}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 y}{\partial x^2}.$$

Setting $c^2 = \frac{T}{\rho}$ we obtain

$$\frac{\partial^2 y}{\partial t^2} = c^2\left(\frac{\partial^2 y}{\partial x^2}\right). \tag{3}$$

Equation (3) is known as the *wave equation*. It is a second order, linear partial differential equation.

It is beyond the scope of these lectures to derive a solution to equation (3). But we can guess some solutions and show that they work. Thus we try a solution of the form

$$y(x, t) = F(x - ct),$$

where F is a continuously differentiable function from $\mathbb{R}$ to itself. Then set $\xi = x - ct$. Then we have, since $\partial\xi/\partial x = 1, \partial\xi/\partial t = -c$,

$$\frac{\partial y}{\partial x} = F'(\xi)\frac{\partial \xi}{\partial x} = F'(\xi),$$

and similarly

$$\frac{\partial^2 y}{\partial x^2} = F''(\xi)\frac{\partial \xi}{\partial x} = F''(\xi),$$

[1] Here we have just used the Taylor expansion $f(x+\epsilon, t) = f(x,t) + \frac{1}{1!}\frac{\partial f}{\partial x}(x,t)\epsilon + \frac{1}{2!}\frac{\partial^2 f}{\partial x^2}(x,t)\epsilon^2 + \cdots$ for small ϵ and set $f = \frac{\partial y}{\partial x}$ and $\epsilon = \delta x/2$.

On the other hand,

$$\frac{\partial y}{\partial t} = F'(\xi)\frac{\partial \xi}{\partial t} = -cF'(\xi),$$

and similarly

$$\frac{\partial^2 y}{\partial t^2} = F''(\xi)\frac{\partial \xi}{\partial t} = c^2 F''(\xi),$$

Hence

$$\frac{\partial^2 y}{\partial t^2} - c^2\left(\frac{\partial^2 y}{\partial x^2}\right) = c^2 F''(\xi) - c^2 F''(\xi) = 0,$$

and so $y(x,t) = F(x-ct)$ is indeed a solution for any continuously differentiable F. It is not difficult to see that for any continuously differentiable G, $y(x,t) = G(x+ct)$ is also a solution. As we have already seen, $F(x-ct)$ is a progressive wave moving to the right with speed c and $G(x+ct)$ is a progressive wave moving to the left with speed c. r Now, there is something very special about *linear* differential equations: If y_1 and y_2 are solutions, then so two is $\alpha_1 y_1 + \alpha_2 y_2$ for any $\alpha_1, \alpha_2 \in \mathbb{R}$. For example,

$$y(x,t) = \alpha F(x-ct) + \beta G(x+ct)$$

is a solution of the wave equation (3) for any $\alpha, \beta \in \mathbb{R}$ (and any continuously differentiable functions F, G).

Simple harmonic waves

These are progressive wave solutions of the wave equation (3) that take the form

$$y(x,t) = A\sin(k(x-ct)), \tag{4}$$

Here c is the speed of the wave and A is the amplitude of the oscillation. The k is known as the wave number. Note that (4) is periodic in both x and t. Fixing an x, the vertical oscillation repeat every period T where

$$y(x,t+T) = A\sin(k(x-ct-cT)) = A\sin(k(x-ct)) = y(x,T),$$

Thus $kcT = 2\pi$, i.e. $T = \dfrac{2\pi}{kc}$ and we sometimes call $\omega = kc$ the angular frequency. For the spatial component, fixing t now,

$$y(x+\lambda,t) = A\sin(k(x+\lambda-ct)) = A\sin(k(x-ct)) = y(x,t),$$

which shows that $k\lambda = 2\pi$, that is $\lambda = \dfrac{2\pi}{k} = cT$. λ is known as the wavelength of the wave. The mode $\sin(k(x-ct))$ is called the first or fundamental harmonic and $\sin(nk(x-ct))$ is the nth harmonic.

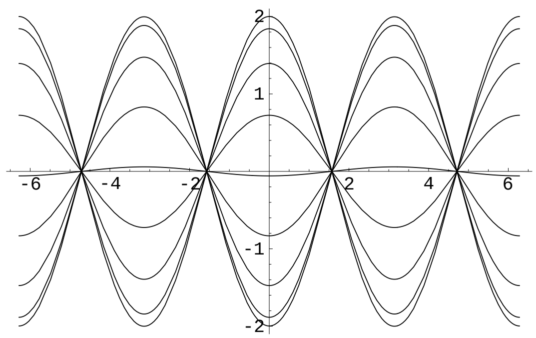

Figure 3: Standing waves formed by two waves of equal amplitude moving with the same speed in opposite directions.

When two identical harmonic waves travelling in opposite directions superpose a standing wave is created. To see this let the waves be $y_1(x,t) = A\cos(k(x-ct))$ and $y_2(x,t) = A\cos(k(x+ct))$. Then when superposed they yield a new wave of the form

$$\begin{aligned} y(x,t) &= y_1(x,t) + y_2(x,t) = A\cos(k(x-ct)) + A\cos(k(x+ct)) \\ &= A(\cos(kx)\cos(kct) + \sin(kx)\sin(kct) + \cos(kx)\cos(kct) - \sin(kx)\sin(kct)) \\ &= 2A\cos(kx)\cos(kct). \end{aligned}$$

Thus for fixed x, say $x = x^*$ the wave oscillates with amplitude $2A\cos(kx^*)$ and frequency kc, i.e. the amplitude of the wave depends on position, but its frequency is the same everywhere along the wave.

For example, at points $x = \dfrac{(2r+1)\pi}{2k}$ (r integer) the wave stays still since here $\cos(kx) = 0$. On the other hand at points $x = \dfrac{r\pi}{k}$ (r integer) the wave moves with maximum amplitude $2A$, since here $|\cos(kx)| = 1$.

Fourier analysis

We may write any solution to (3) as a Fourier series. Let us illustrate this by example. Suppose we take a string length ℓ, and fix it a both ends. When we pluck it we get standing waves produced by progressive waves travelling in each direction being reflected back by the fixed boundaries. These standing waves must satisfy the boundary conditions: $y(0,t) = 0 = y(\ell,t)$. The standing waves

$$\sin(kx)(A\cos(kct) + B\sin(kct))$$

satisfy the wave equation (3) and also the boundary conditions, for each $A, B \in \mathbb{R}$, provided that

$$k\ell = n\pi,$$

where n is any integer (since $\sin(kx) = \sin(n\pi x/\ell) = 0$ when $x = 0, \ell$). Thus

$$y_n(x,t) = \sin(\frac{n\pi x}{\ell})\left(A_n\cos(\frac{n\pi ct}{\ell}) + B_n\sin(\frac{n\pi ct}{\ell})\right)$$

are all standing wave solutions of (3) for each integer n *that satisfy the boundary conditions that says that the string is fixed at* $x = 0, \ell$. We look for the displacement of the string expressed as a sum

$$y(x,t) = \sum_{n=1}^{\infty} \left\{ A_n \sin(\frac{n\pi x}{\ell}) \cos(\frac{n\pi ct}{\ell}) + B_n \sin(\frac{n\pi x}{\ell}) \sin(\frac{n\pi ct}{\ell}) \right\}.$$

The coefficients A_n, B_n are found by setting $t = 0$ and using the initial conditions:

$$y(x,0) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{\ell}), \tag{5}$$

where $y(x,0)$ is the shape of the plucked string before being released, and

$$\frac{\partial y}{\partial t}(x,0) = v(x) = 0, \tag{6}$$

since the string is released from rest. We will now need the following identity:

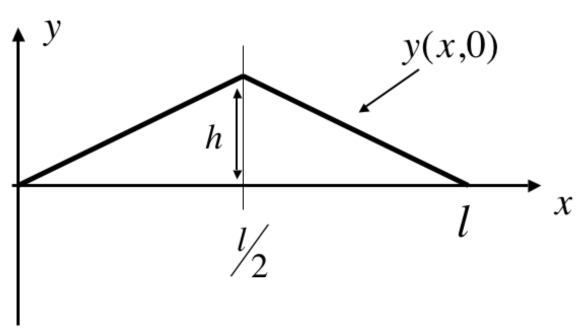

Figure 4: Initial displacement of a plucked string

$$\int_0^{\ell} \sin(\frac{n\pi x}{\ell}) \sin(\frac{m\pi x}{\ell})\, dx = \delta_{mn} \frac{\ell}{2} \tag{7}$$

where δ_{mn} is the Kronecker-delta function ($\delta_{mn} = 0$ if $m \neq n$ and $\delta_{nn} = 1$.) To prove (7) simply write, for $n \neq m$,

$$\begin{aligned} \int_0^{\ell} \sin(\frac{n\pi x}{\ell}) \sin(\frac{m\pi x}{\ell})\, dx &= \frac{1}{2} \int_0^{\ell} \cos\left(\frac{(n-m)\pi x}{\ell}\right) - \cos\left(\frac{(m+n)\pi x}{\ell}\right) dx \\ &= \frac{1}{2} \left[\frac{\ell}{\pi(n-m)} \sin\left(\frac{(n-m)\pi x}{\ell}\right) - \frac{\ell}{\pi(m+n)} \sin\left(\frac{(m+n)\pi x}{\ell}\right) \right]_0^{\ell} \\ &= 0, \end{aligned}$$

whereas when $n = m$ we have

$$\int_0^\ell \sin^2(\frac{n\pi x}{\ell})\, dx = \int_0^\ell \frac{1}{2}\left(1 - \cos(\frac{2n\pi x}{\ell})\right)\, dx = \ell/2.$$

To find A_n we need to solve (5). We find that

$$A_n = \frac{2}{\ell}\int_0^\ell y(x,0)\sin(\frac{n\pi x}{\ell})\, dx.$$

To see this, multiply by $\sin(\frac{m\pi x}{\ell})$ and integrate:

$$\begin{aligned}\int_0^\ell y(x,0)\sin(\frac{m\pi x}{\ell})\, dx &= \int_0^\ell \sum_{n=1}^\infty A_n \sin(\frac{n\pi x}{\ell})\sin(\frac{m\pi x}{\ell})\, dx \\ &= \sum_{n=1}^\infty A_n \left\{\int_0^\ell \sin(\frac{n\pi x}{\ell})\sin(\frac{m\pi x}{\ell})\, dx\right\} \\ &= A_m \frac{\ell}{2}\end{aligned}$$

using (7) (since the only non-zero term in the series is when $m = n$), which gives the desired formula for A_m.

From (6) we find that

$$v(x) = \frac{\partial y}{\partial t}(x,0) = \sum_{n=1}^\infty B_n \frac{n\pi c}{\ell}\sin(\frac{n\pi x}{\ell})$$

Thus

$$\int_0^\ell v(x)\sin(\frac{m\pi x}{\ell})\, dx = \int_0^\ell \sum_{n=1}^\infty B_n \frac{n\pi c}{\ell}\sin(\frac{n\pi x}{\ell})\sin(\frac{m\pi x}{\ell})\, dx$$

This then gives, since all terms except $m = n$ in the sum vanish,

$$B_n = \frac{2}{n\pi c}\int_0^\ell v(x)\sin(\frac{n\pi x}{\ell})\, dx = 0$$

for all n. For the function shown

$$y(x,0) = \begin{cases} \dfrac{2hx}{\ell} & x \in [0, \dfrac{\ell}{2}) \\ 2h\left(1 - \dfrac{x}{\ell}\right) & x \in [\dfrac{\ell}{2}, \ell], \end{cases}$$

we find that

$$A_n = \begin{cases} \frac{8h}{\pi^2 n^2}\sin(n\pi/2) & n \text{ odd} \\ 0 & n \text{ even.} \end{cases}$$

We can do this calculation by splitting the integral

$$I = \int_0^\ell y(x,0)\sin(\frac{n\pi x}{\ell})\, dx = \int_0^{\frac{\ell}{2}} \frac{2hx}{\ell}\sin(\frac{n\pi x}{\ell})\, dx + \int_{\frac{\ell}{2}}^\ell 2h\left(1 - \frac{x}{\ell}\right)\sin(\frac{n\pi x}{\ell})\, dx.$$

Now

$$\begin{aligned}
\int_0^{\frac{\ell}{2}} x\sin(\frac{n\pi x}{\ell})\,dx &= \left[-x\frac{\ell}{n\pi}\cos(\frac{n\pi x}{\ell})\right]_0^{\ell/2} + \int_0^{\ell/2} 1.\frac{\ell}{n\pi}\cos(\frac{n\pi x}{\ell})\,dx \\
&= -\frac{\ell^2}{2n\pi}\cos(n\pi/2) + \frac{\ell^2}{n^2\pi^2}\left[\sin(\frac{n\pi x}{\ell})\right]_0^{\ell/2} \\
&= -\frac{\ell^2}{2n\pi}\cos(n\pi/2) + \frac{\ell^2}{n^2\pi^2}\sin(\frac{n\pi}{2}).
\end{aligned}$$

Also we have

$$\begin{aligned}
\int_{\frac{\ell}{2}}^{\ell}\left(1-\frac{x}{\ell}\right)\sin(\frac{n\pi x}{\ell})\,dx &= \left[-\left(1-\frac{x}{\ell}\right)\frac{\ell}{n\pi}\cos(\frac{n\pi x}{\ell})\right]_{\ell/2}^{\ell} - \int_{\ell/2}^{\ell}(-\frac{1}{\ell})(-\frac{\ell}{n\pi})\cos(\frac{n\pi x}{\ell})\,dx \\
&= \frac{\ell}{2n\pi}\cos(\frac{n\pi}{2}) - \frac{1}{n\pi}\left[\frac{\ell}{n\pi}\sin(\frac{n\pi x}{\ell})\right]_{\ell/2}^{\ell} \\
&= \frac{\ell}{2n\pi}\cos(\frac{n\pi}{2}) + \frac{\ell}{n^2\pi^2}\sin(\frac{n\pi}{2}).
\end{aligned}$$

This gives

$$\begin{aligned}
I &= \frac{2h}{\ell}\left(-\frac{\ell^2}{2n\pi}\cos(\frac{n\pi}{2}) + \frac{\ell^2}{n^2\pi^2}\sin(\frac{n\pi}{2})\right) + 2h\left(\frac{\ell}{2n\pi}\cos(\frac{n\pi}{2}) + \frac{\ell}{n^2\pi^2}\sin(\frac{n\pi}{2})\right) \\
&= \frac{4h\ell}{n^2\pi^2}\sin(\frac{n\pi}{2}).
\end{aligned}$$

Since $A_n = \frac{2}{\ell}I$ we get the desired value of A_n.

Thus we finally obtain, since $\sin(n\pi/2) = 0$ if n is even and $(-1)^{n-1}$ if n is odd, the displacement

$$y(x,t) = \frac{8h}{\pi^2}\sum_{r=0}^{\infty}\frac{(-1)^r}{(2r+1)^2}\sin\left(\frac{(2r+1)\pi x}{\ell}\right)\cos\left(\frac{(2r+1)\pi ct}{\ell}\right).$$

Notice that the factor of $(2r+1)^{-2}$ in the sum means that the main contribution comes from the first few harmonics.

The phenomenon of Beats

Beats occurs when two waves of nearby frequencies superpose. If two tuning forks of nearby frequencies are knocked one hears a continuous tone close to the tuning fork frequencies that attenuates slowly with time. The resulting phenomenon is known as beats.

Let's first look at the simple case where two (e.g. sound) waves have equal amplitudes and speed c, but slightly different frequencies and wavelengths, e.g. $y_1(x,t) = A\cos(k_1(x-ct))$ and

$y_2(x,t) = A\cos(k_2(x-ct))$ where $k_2 = k_1 + \epsilon$ with ϵ small. Then

$$\begin{aligned} y_1(x,t) + y_2(x,t) &= A\cos(k_1(x-ct)) + A\cos(k_2(x-ct)) \\ &= 2A\cos\left(\frac{(k_1+k_2)}{2}(x-ct)\right)\cos\left(\frac{(k_1-k_2)}{2}(x-ct)\right) \\ &= 2A\cos\left(\frac{(2k_1+\epsilon)}{2}(x-ct)\right)\cos\left(\frac{\epsilon}{2}(x-ct)\right) \end{aligned}$$

This wave is shown in figure 5. It is composed of a slow envelope variation of frequency $c\epsilon/2$ and a faster oscillation of frequency $(k_1 + \epsilon/2)c$, and so whose frequency is barely distinguishable from that of the original individual waves. A person listening to such a wave at some fixed location $x = x^*$ will thus hear very close to the correct frequency, but at an amplitude that varies slowly at the timescale of ϵ.

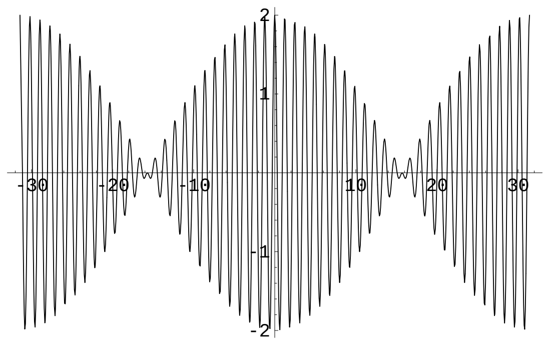

Figure 5: Beats: superposition of two sound waves of nearby frequencies. The figure shows amplitude versus time at a fixed position in space, e.g. where an observer hears the sound. The resulting waveform is composed of a slowly varying envelope of frequency $\epsilon/2$ and a second oscillation within the envelope of frequency close to the original waves.

Beats also occur in the forced oscillator model when ω and μ are close, say $\mu = \omega + \epsilon$ with ϵ small. Recall that from the previous week's lectures on Oscillations that the amplitude of a forced simple harmonic oscillator is

$$Y(t) = A\cos(\omega t + \delta) + \frac{f_0}{m(\omega^2 - \mu^2)}\cos(\mu t).$$

Let $B = \frac{f_0}{m(\omega^2-\mu^2)}$. Then

$$
\begin{aligned}
Y(t) &= A\cos(\omega t+\delta) + B\cos(\mu t) \\
&= A\cos(\omega t+\delta) + B\cos(\omega t+\epsilon t) \\
&= A\cos(\omega t+\delta) + B\cos(\omega t+\delta+\epsilon t-\delta) \\
&= A\cos(\Phi(t)) + B\cos(\Phi(t)+\phi(t)) \quad \text{where } \Phi(t) = \omega t+\delta, \phi(t) = \epsilon t-\delta \\
&= A\cos(\Phi(t)) + B(\cos(\Phi(t))\cos(\phi(t)) - \sin(\Phi(t))\sin(\phi(t)) \\
&= (A + B\cos(\phi(t)))\cos(\Phi(t)) - B\sin(\Phi(t))\sin(\phi(t)) \\
&= \sqrt{(A+B\cos(\phi(t)))^2 + B^2\sin(\phi(t))^2} \times \\
&\left\{\frac{A+B\cos(\phi(t))}{\sqrt{(A+B\cos(\phi(t)))^2 + B^2\sin(\phi(t))^2}}\cos(\Phi(t)) - \frac{B\sin(\phi(t))}{\sqrt{(A+B\cos(\phi(t)))^2 + B^2\sin(\phi(t))^2}}\sin(\Phi(t))\right\} \\
&= \Gamma(t)\cos(\Phi(t)+\theta(t)),
\end{aligned}
$$

where

$$\Gamma(t) = \sqrt{A^2+B^2+2AB\cos(\phi(t))},$$

and $\theta(t)$ is given by

$$\tan(\theta(t)) = \frac{B\sin(\phi(t))}{A+B\cos(\phi(t))}.$$

Now $\phi(t) = \epsilon t - \delta$ is very slowly changing in t since ϵ is small, so that $\Gamma(t)$ and $\theta(t)$ are also very slowly changing with t. $\Phi(t)$ on the other hand changes at the timescale of the natural frequency ω. Thus we see that $Y(t) = \Gamma(t)\cos(\Phi(t) + \theta(t))$ is composed of a slowly evolving envelope with amplitude $\Gamma(t)$ within which there is a oscillation at a frequency approximately equal to the natural frequency ω, i.e. we have beats.