Math 1302, Week 8: Oscillations

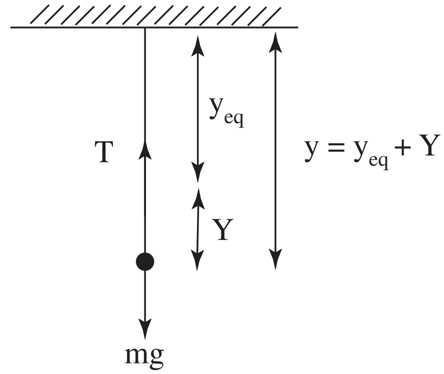

Figure 1: Simple harmonic motion. At equilibrium the string is of total length y_{eq}. During the motion we let Y be the extension beyond equilibrium, so that the total length is $y = y_{eq} + Y$.

Recall vertical oscillations of a mass on a light elastic string (see figure 1). Let y be the downward displacement of the mass m from the point O where the string is fixed. Then we have $m\ddot{y} = mg - T$ where T is the tension in the string, given by $T = k(y - \ell)$, where ℓ is the natural length of the string (i.e. its unstretched length). Thus $m\ddot{y} = mg - k(y - \ell)$. Now if the string is resting in equilibrium, the weight equals the tension, so if y_{eq} is the equilibrium value of y, $T_{eq} = k(y_{eq} - \ell) = mg$, which on rearrangement gives $y_{eq} = \frac{mg}{k} + \ell$. Now suppose that the mass is pulled down and released. We will assume that the string remains taut for all t (which is so if it is not stretched further than $y_{eq} - \ell$ beyond equilibrium). Let Y be the extension of the string beyond equilibrium: $y = y_{eq} + Y$. Then the equation of motion becomes

$$m\frac{d^2}{dt^2}(y_{eq} + Y) = mg - k(y_{eq} + Y - \ell),$$

Since y_{eq} is constant,

$$m\ddot{Y} = mg - k(y_{eq} - \ell) - kY = -kY,$$

since the equilibrium y_{eq} satisfies $k(y_{eq} - \ell) = mg$. The general solution of $\ddot{Y} = -\frac{k}{m}Y$ is

$$Y(t) = A\cos(\omega t + \delta),$$

where A, δ are constants determined by the initial conditions and $\omega^2 = k/m$. Here A is the amplitude of the oscillation, ω is its angular frequency and we have that its period $T = 2\pi/\omega$.

Another way of approaching this problem is to note that the tension $T = T_{eq} + kY$, since the stretch of the string Y beyond the equilibrium tension T_{eq} provides an additional force of kY. Newton's 2nd law then gives

$$m\ddot{Y} = mg - T = mg - T_{eq} - kY = (mg - T_{eq}) - kY = -kY,$$

since at equilibrium $T_{eq} = mg$.

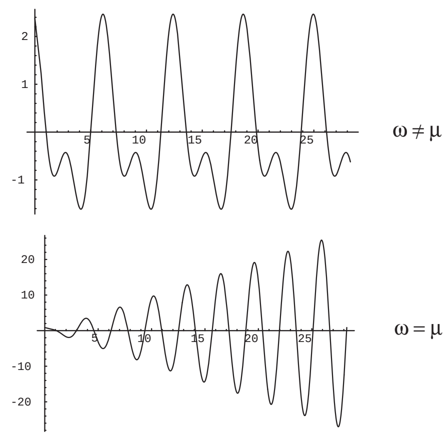

Figure 2: Forced simple harmonic oscillator. When the forcing frequency μ is distinct from the oscillator frequency ω the system remains bounded. When $\mu = \omega$ we obtain resonance in which the amplitude of the oscillation increases with t.

Forced oscillations

Suppose that we add a periodic component of forcing to the mass[1], say $f(t) = f_0 \cos(\mu t)$. Then the equations of motion (for when the string remains taut and there are no collisions between the mass and the other objects) become

$$\ddot{Y} + \frac{k}{m} Y = \frac{f(t)}{m} = \frac{f_0}{m} \cos(\mu t). \tag{1}$$

Recall that to solve such linear second order differential equations, we first solve the homogeneous equation $\ddot{Y} + \frac{k}{m} Y = 0$ for the complementary functions and then add on the particular integral. Thus with $\omega^2 = k/m$ we get

$$Y(t) = A \cos(\omega t + \delta) + Y_{PI}.$$

Now if $\mu \neq \omega$ we may try a PI of the form $Y_{PI} = C \cos(\mu t)$, which gives

$$\frac{f_0}{m} \cos(\mu t) = -\mu^2 C \cos(\mu t) + \omega^2 C \cos(\mu t),$$

so that $C = \frac{f_0}{m(\omega^2 - \mu^2)}$ and general solution is

$$Y(t) = A \cos(\omega t + \delta) + \frac{f_0}{m(\omega^2 - \mu^2)} \cos(\mu t). \tag{2}$$

On the other hand, when $\mu = \omega$ the particular integral becomes $Y_{PI} = B' t \sin(\omega t)$. Substituting into the differential equation (1) we obtain

$$\frac{f_0}{m} \cos(\omega t) = 2\omega B' \cos(\omega t) - B' \omega^2 t \sin(\omega t) + B' \omega^2 t \sin(\omega t) = 2\omega B' \cos(\omega t)$$

[1]For example, if the mass carries a charge, we could impose a periodic electric field.

so that $B' = \frac{f_0}{2m\omega}$. Thus in this case we have

$$Y(t) = A\cos(\omega t + \delta) + \frac{f_0}{2m\omega} t \sin(\omega t).$$

Note that after a certain time the oscillations have grown so large that the string does not remain taut during the motion and the solution no longer applies.

Compound oscillators

Now consider a system of two masses M, m on two elastic strings as follows. Mass M is suspended from O on a string natural length ℓ_1 and with constant k_1. The mass M is attached to the mass m by a string of natural length ℓ_2 and constant k_2. We let Y_1 be the distance of m below its equilibrium, and Y_2 the distance of M below its equilibrium (see figure 3). As usual we will

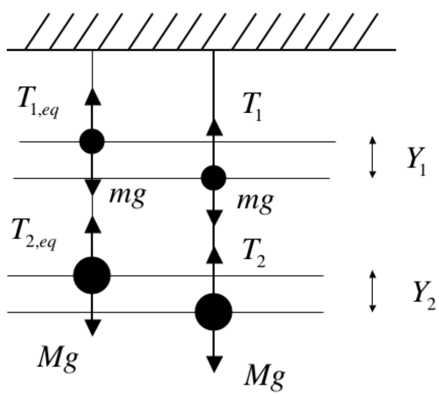

Figure 3: Compound pendulum of two mass with two elastic strings. For $i = 1, 2$, Y_i is the extension of string i beyond equilibrium.

assume that the motion is such that the strings remain taut for all time. The equations of motion are, using Newton's second law,

$$\begin{aligned} m\ddot{Y}_1 &= T_2 + mg - T_1 \\ M\ddot{Y}_2 &= Mg - T_2 \end{aligned}$$

where

$$T_1 = T_{1,eq} + k_1 Y_1, \quad T_2 = T_{2,eq} + k_2(Y_2 - Y_1).$$

In equilibrium

$$T_{1,eq} = T_{2,eq} + mg, \quad T_{2,eq} = Mg.$$

Thus we have

$$\begin{aligned} m\ddot{Y}_1 &= T_2 + mg - T_1 \\ &= T_{2,eq} + k_2(Y_2 - Y_1) + mg - T_{1,eq} - k_1 Y_1 \\ &= T_{2,eq} + mg - T_{1,eq} + k_2(Y_2 - Y_1) - k_1 Y_1 \\ &= k_2(Y_2 - Y_1) - k_1 Y_1, \end{aligned}$$

since $T_{2,eq} + mg - T_{1,eq} = 0$. Similarly,

$$\begin{aligned} M\ddot{Y}_2 &= Mg - T_2 \\ &= Mg - T_{2,eq} - k_2(Y_2 - Y_1) \\ &= -k_2(Y_2 - Y_1), \end{aligned}$$

since $Mg - T_{2,eq} = 0$.

To summarize:

$$\begin{aligned} m\ddot{Y}_1 &= -(k_1 + k_2)Y_1 + k_2 Y_2 \\ M\ddot{Y}_2 &= k_2 Y_1 - k_2 Y_2. \end{aligned} \tag{3}$$

We seek solutions of this system of the form

$$\underline{Y} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \cos(\omega t + \delta). \tag{4}$$

For such a solution we have $\ddot{Y}_1 = -\omega^2 C_1 \cos(\omega t + \delta)$ and $\ddot{Y}_2 = -\omega^2 C_2 \cos(\omega t + \delta)$. Thus if (4) is a solution of (3)

$$\begin{aligned} -\omega^2 m C_1 \cos(\omega t + \delta) &= -(k_1 + k_2) C_1 \cos(\omega t + \delta) + k_2 C_2 \cos(\omega t + \delta) \\ -\omega^2 M C_2 \cos(\omega t + \delta) &= k_2 C_1 \cos(\omega t + \delta) - k_2 C_2 \cos(\omega t + \delta). \end{aligned}$$

Since this is true for all t, we have, rearranging

$$\begin{aligned} (m\omega^2 - k_1 - k_2)C_1 + k_2 C_2 &= 0 \\ k_2 C_1 + (M\omega^2 - k_2)C_2 &= 0. \end{aligned}$$

If this is to provide a non-trivial solution $\underline{Y} \neq \underline{0}$, i.e. $(C_1, C_2)^T \neq \underline{0}$ we must have

$$(m\omega^2 - k_1 - k_2)(M\omega^2 - k_2) - k_2^2 = 0.$$

Set $q = \omega^2$. Then we obtain the following quadratic for the possible values of q:

$$Mmq^2 - (k_2 m + M(k_1 + k_2))q + k_1 k_2 = 0.$$

This has two roots; are they real? They are if

$$(k_2 m + (k_1 + k_2)M)^2 \geq 4Mmk_1 k_2.$$

But $k_2m + (k_1+k_2)M \geq k_2m + k_1M$ so that

$$(k_2m + (k_1+k_2)M)^2 - 4Mmk_1k_2 \geq (k_2m + k_1M)^2 - 4Mmk_1k_2 = (k_2m - k_1M)^2 \geq 0,$$

so the roots are real. Lets call them q_-, q_+. Since $k_1k_2 > 0$, we see that both real roots are positive, by noting that q satisfies

$$q = \frac{k_1k_2 + Mmq^2}{k_2m + M(k_1+k_2)}.$$

Since we have show that q is real, the right-hand side is positive, and hence both $q_-, q_+ > 0$. Thus the four possible values of ω, namely $\pm\sqrt{q_+}, \pm\sqrt{q_-}$ are all real. We only need only concern ourselves with ω_+^2, ω_-^2.

Now we may find the possible C_1, C_2 for each value of $q = \omega^2$. When $\omega^2 = q_-$ we have

$$C_1k_2 = C_2(k_2 - M\omega^2) = C_2(k_2 - Mq_-).$$

Hence one possible solution is, for any real B_-,

$$\underline{Y}_- = B_- \begin{pmatrix} \frac{k_2 - Mq_-}{k_2} \\ 1 \end{pmatrix} \cos(\sqrt{q_-}t + \delta_-).$$

When $\omega^2 = q_+$ we have

$$C_1k_2 = C_2(k_2 - M\omega^2) = C_2(k_2 - Mq_+),$$

giving another possible solution, for any real B_+,

$$\underline{Y}_+ = B_+ \begin{pmatrix} \frac{k_2 - Mq_+}{k_2} \\ 1 \end{pmatrix} \cos(\sqrt{q_+}t + \delta_+).$$

The general solution is any linear combination of these two solutions: for any real α, β

$$\underline{Y}(t) = \alpha \begin{pmatrix} \frac{k_2 - Mq_-}{k_2} \\ 1 \end{pmatrix} \cos(\sqrt{q_-}t + \delta_-) + \beta \begin{pmatrix} \frac{k_2 - Mq_+}{k_2} \\ 1 \end{pmatrix} \cos(\sqrt{q_+}t + \delta_+)$$

The frequencies $\omega_- = \sqrt{q_-}, \omega_+ = \sqrt{q_+}$ are called the <u>eigenfrequencies</u> of the system, and the $\underline{Y}_-$, $\underline{Y}_+$ are the respective <u>normal modes</u> associated with these frequencies.

Illustrative example

Suppose we have $M = 2, m = 3$ and $k_1 = k_2 = 1$. Then we have

$$\begin{aligned} 3\ddot{Y}_1 &= -2Y_1 + Y_2 \\ 2\ddot{Y}_2 &= Y_1 - Y_2 \end{aligned}$$

Trying a solution

$$\underline{Y} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} \cos(\omega t + \delta)$$

we obtain

$$-3\omega^2 A = -2A + B, \quad A + (2\omega^2 - 1)B = 0,$$

Setting $q = \omega^2$ we obtain $6q^2 - 7q + 1 = 0$, so $q = 1, 1/6$. This gives the eigenfrequencies $\omega_- = 1/\sqrt{6}, \omega_+ = 1$.

For $\omega = 1$ we obtain $A = -B$ and so the normal modes for w_+ are any multiple of

$$\underline{Y}_+ = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(t + \delta_+).$$

For $\omega = 1/\sqrt{6}$ we find that $3A = 2B$ and hence the normal modes for w_- are any multiple of

$$\underline{Y}_- = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \cos(\frac{t}{\sqrt{6}} + \delta_-).$$

The general solution is thus, for any real α, β,

$$\underline{Y}(t) = \alpha \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(t + \delta_+) + \beta \begin{pmatrix} 2 \\ 3 \end{pmatrix} \cos(\frac{t}{\sqrt{6}} + \delta_-).$$

For the normal mode $\underline{Y}(t) = \underline{Y}_-(t)$ we see that $Y_1(t) = -Y_2(t)$ for all t. This means that in this mode the particles oscillate in antiphase (figure 4). For $q = q_- = 1/\sqrt{6}$, the particles oscillate

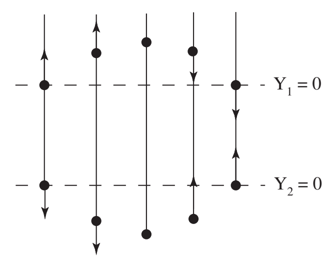

Figure 4: Normal mode $q = q_+ = 1$ for the compound pendulum of two mass with two elastic strings where particle oscillate in antiphase: $Y_1 = -Y_2$

in phase, but the amplitude of Y_2 is one and a half times that of Y_1 (figure 5).

In figure 6 we show typical plots of these two modes and the general solution.

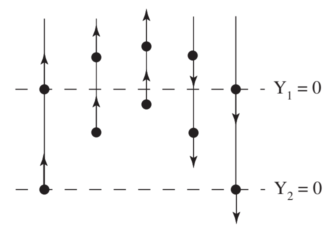

Figure 5: Normal mode $q = q_- = 1/\sqrt{6}$ for the compound pendulum of two mass with two elastic strings where particle oscillate in phase, but with different amplitudes $Y_1 = \frac{2}{3}Y_2$

Example: Two masses joined by three springs

Consider the system of two identical masses of mass m linked by three identical springs of stiffness k (see figure 7). The equations of motion are

$$m\ddot{x}_1 = T_2 - T_1 \quad (5)$$
$$m\ddot{x}_2 = T_3 - T_2, \quad (6)$$

where $T_1 = T_{eq} + kx_1$, $T_2 = T_{eq} + k(x_2 - x_1)$ and $T_3 = T_{eq} - kx_2$. Thus we obtain

$$m\ddot{x}_1 = -2kx_1 + kx_2 \quad (7)$$
$$m\ddot{x}_2 = kx_1 - 2kx_2, \quad (8)$$

This may be rewritten as

$$\ddot{\underline{x}} = -A\underline{x}, \quad (9)$$

where $\underline{x} = (x_1, x_2)^T$ and

$$A = \begin{pmatrix} 2\sigma & -\sigma \\ -\sigma & 2\sigma \end{pmatrix}.$$

Now the eigenvalues of A are σ and 3σ: They are the roots of

$$\det \begin{pmatrix} 2\sigma - \lambda & -\sigma \\ -\sigma & 2\sigma - \lambda \end{pmatrix} = \lambda^2 - 4\sigma\lambda + 3\sigma^2 = 0.$$

For $\lambda = 3\sigma$ we may take an eigenvector $\underline{v}_1 = (1, -1)^T$ and for $\lambda = \sigma$ we may take $\underline{v}_2 = (1, 1)^T$. Now note that if

$$P = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},$$

then

$$P^{-1} = \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix},$$

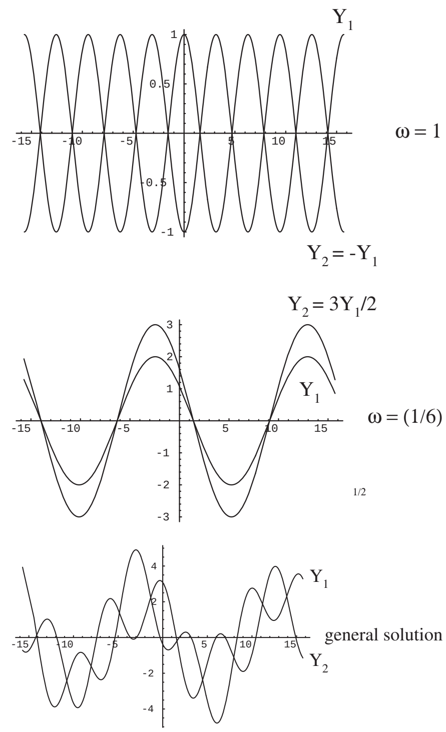

Figure 6: First plot: Normal mode $q = q_+ = 1$, antiphase; Second plot: Normal mode $q = q_- = 1/\sqrt{6}$, in phase. Third plot, a typical solution constructed from combinations of the two modes.

and $P^{-1}AP = \text{diag}(3\sigma, \sigma) = D$. Thus from (9) we have,

$$\ddot{\underline{x}} = -PDP^{-1}\underline{x},$$

and so

$$P^{-1}\ddot{\underline{x}} = -DP^{-1}\underline{x}.$$

Now define $\underline{Y} = (Y_1, Y_2)^T = P^{-1}\underline{x}$, to obtain

$$\ddot{\underline{Y}} = -D\underline{Y},$$

which expands to

$$\begin{aligned} \ddot{Y}_1 &= -3\sigma Y_1 \\ \ddot{Y}_2 &= -\sigma Y_2. \end{aligned}$$

Thus

$$Y_1 = \alpha_1 \cos(\omega_1 t + \delta_1), \quad Y_2 = \alpha_2 \cos(\omega_2 t + \delta_2),$$

where $\alpha_1, \alpha_2, \delta_1, \delta_2$ are constants and $\omega_1 = \sqrt{3\sigma}, \omega_2 = \sqrt{\sigma}$. Now we use $\underline{x} = P\underline{Y}$ to obtain

$$\begin{aligned} x_1 &= Y_1 + Y_2 = \alpha_1 \cos(\omega_1 t + \delta_1) + \alpha_2 \cos(\omega_2 t + \delta_2) \\ x_2 &= -Y_1 + Y_2 = -\alpha_1 \cos(\omega_1 t + \delta_1) + \alpha_2 \cos(\omega_2 t + \delta_2). \end{aligned}$$

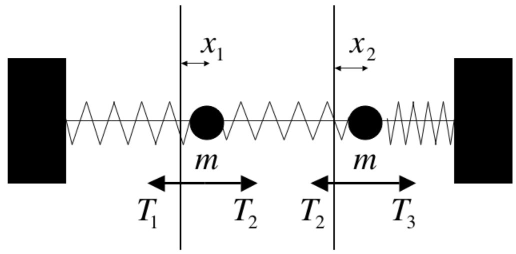

Figure 7: Two identical masses between 3 identical springs

Notice that we may also write this as

$$\underline{x} = \underline{v}_1 \alpha_1 \cos(\omega_1 t + \delta_1) + \underline{v}_2 \alpha_2 \cos(\omega_2 t + \delta_2).$$

Such a decomposition works in general: Suppose an oscillator system has $\underline{x} = (x_1, \ldots, x_n)^T$ and satisfies

$$\ddot{\underline{x}} = -A\underline{x}, \tag{10}$$

where A is a real $n \times n$ matrix with distinct eigenvalues then the general solution is

$$\underline{x}(t) = \sum_{k=1}^{n} \alpha_k \underline{v}_k \cos(\sqrt{\lambda_k} t + \delta_k),$$

where $\underline{v}_k$ is an eigenvector associated with the eigenvalue λ_k of A and the α_k, δ_k are constants to be found from the initial conditions.

To prove this, we diagonalise A as above: $P^{-1}AP = D$ where $D = \text{diag}(\lambda_1, \ldots, \lambda_n)$ and P is made up from eigenvectors as columns. From (10) we obtain

$$\ddot{\underline{x}} = -PDP^{-1}\underline{x},$$

so that as above we set $\underline{Y} = P^{-1}\underline{x}$ to obtain $P^{-1}\ddot{\underline{x}} = -P^{-1}PDP^{-1}\underline{x} = -DP^{-1}\underline{x}$ so that $\ddot{\underline{Y}} = -D\underline{Y}$. This gives $\ddot{Y}_k = -\lambda_k Y_k$ for $k = 1, \ldots, n$ which has general solution $Y_k(t) = \alpha_k \cos(\sqrt{\lambda_k} t + \delta_k)$ for $k = 1, \ldots, n$. Then we finally have

$$\underline{x} = P\underline{Y} = \underbrace{(\underline{v}_1 | \underline{v}_2 |, \cdots, | \underline{v}_n)}_{\text{matrix of column vectors } \underline{v}_k} \begin{pmatrix} \alpha_1 \cos(\sqrt{\lambda_1} t + \delta_1) \\ \alpha_2 \cos(\sqrt{\lambda_2} t + \delta_2) \\ \vdots \\ \alpha_n \cos(\sqrt{\lambda_n} t + \delta_n) \end{pmatrix} = \sum_{k=1}^{n} \alpha_k \underline{v}_k \cos(\sqrt{\lambda_k} t + \delta_k).$$

Example: Compound pendulum

Consider the compound pendulum shown in figure 8. Two masses, mass m, M, are supported by two light inextensible strings both of length a. We will assume that the strings remain taut for all time. From Newton's 2nd law for mass m:

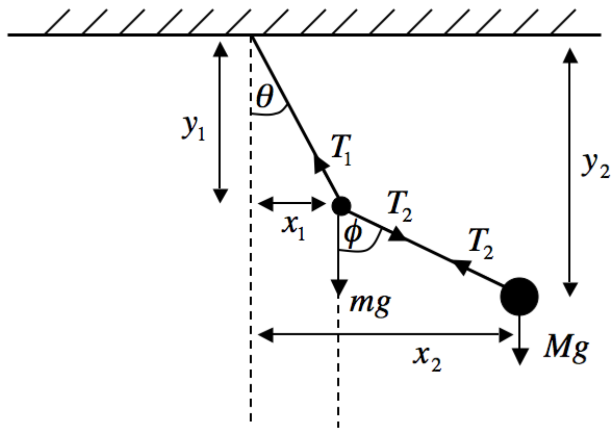

Figure 8: Compound pendulum: two masses on light inextensible strings length a swinging in the vertical plane.

$$
\begin{aligned}
m\ddot{x}_1 &= T_2 \sin\phi - T_1 \sin\theta \\
m\ddot{y}_1 &= mg + T_2 \cos\phi - T_1 \cos\theta.
\end{aligned}
$$

Similarly, for the mass M,

$$
\begin{aligned}
M\ddot{x}_2 &= -T_2 \sin\phi \\
M\ddot{y}_2 &= Mg - T_2 \cos\phi.
\end{aligned}
$$

Geometrically,

$$x_1 = a\sin\theta,\ y_1 = a\cos\theta,\ x_2 = x_1 + a\sin\phi,\ y_2 = y_1 + a\cos\phi.$$

We assume that oscillations are small: θ, ϕ remain small and $\sin\theta \approx \theta$, $\cos\theta \approx 1$ and similarly for the angle ϕ. This gives

$$x_1 \approx a\theta,\ y_1 \approx a,\ x_2 \approx a(\theta + \phi),\ y_2 \approx 2a.$$

Putting these approximations into the above equations of motion yields, the approximation for small angles:

$$
\begin{aligned}
ma\ddot{\theta} &= T_2\phi - T_1\theta & (11)\\
0 &= mg + T_2 - T_1 & (12)\\
Ma(\ddot{\theta} + \ddot{\phi}) &= -T_2\phi & (13)\\
0 &= Mg - T_2. & (14)
\end{aligned}
$$

From (12) and (14) we obtain (for this approximation)

$$T_2 = Mg,\ T_1 = Mg + mg.$$

Hence from (11) and (13) we have

$$
\begin{aligned}
ma\ddot{\theta} &= Mg\phi - (M+m)g\theta & (15)\\
Ma(\ddot{\theta} + \ddot{\phi}) &= -Mg\phi. & (16)
\end{aligned}
$$

We may rewrite this as

$$\begin{pmatrix} ma & 0 \\ Ma & Ma \end{pmatrix} \frac{d^2}{dt^2} \begin{pmatrix} \theta \\ \phi \end{pmatrix} = \begin{pmatrix} -(M+m)g & Mg \\ 0 & -Mg \end{pmatrix} \begin{pmatrix} \theta \\ \phi \end{pmatrix}$$

Hence

$$\frac{d^2}{dt^2} \begin{pmatrix} \theta \\ \phi \end{pmatrix} = \begin{pmatrix} ma & 0 \\ Ma & Ma \end{pmatrix}^{-1} \begin{pmatrix} -(M+m)g & Mg \\ 0 & -Mg \end{pmatrix} \begin{pmatrix} \theta \\ \phi \end{pmatrix} = -A \begin{pmatrix} \theta \\ \phi \end{pmatrix}$$

where

$$A = \begin{pmatrix} \frac{g}{ma}(M+m) & -\frac{Mg}{ma} \\ -\frac{g}{ma}(M+m) & \frac{g}{ma}(M+m) \end{pmatrix}$$

The characteristic equation for the eigenvalues λ of A reads

$$\lambda^2 - \frac{2g}{ma}(M+m)\lambda + \left(\frac{g}{ma}(M+m)\right)^2 - \frac{Mg}{ma}\left(\frac{g}{ma}(M+m)\right) = 0,$$

that is:

$$\lambda^2 - \frac{2g}{ma}(M+m)\lambda + \frac{g^2}{ma^2}(M+m) = 0.$$

This gives two real and positive solutions

$$\lambda_\pm = \frac{g}{ma}\left((m+M) \pm \sqrt{M(m+M)}\right),$$

and so the eigenfrequencies are

$$\omega_\pm = \sqrt{\frac{g}{ma}\left((m+M) \pm \sqrt{M(m+M)}\right)}.$$

When $m = M$ we obtain

$$\omega_{\pm} = \sqrt{\frac{2g \pm \sqrt{2}g}{a}}.$$

For $\omega = \omega_+$ we have an eigenvector $(1, -1/\sqrt{2})^T$ and when $\omega = \omega_-$ we have the eigenvector $(1, 1/\sqrt{2})^T$. Hence the two normal modes are

$$\begin{pmatrix} \theta \\ \phi \end{pmatrix}_{\pm} = \begin{pmatrix} 1 \\ \mp\frac{1}{\sqrt{2}} \end{pmatrix} \cos\left(\left(\frac{2g \pm \sqrt{2}g}{a}\right)^{\frac{1}{2}} t + \delta_{\pm}\right).$$

Notice that the angular velocities of these normal modes are

$$\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}_{\pm} = \left(\frac{2g \pm \sqrt{2}g}{a}\right)^{\frac{1}{2}} \begin{pmatrix} -1 \\ \pm\frac{1}{\sqrt{2}} \end{pmatrix} \sin\left(\left(\frac{2g \pm \sqrt{2}g}{a}\right)^{\frac{1}{2}} t + \delta_{\pm}\right),$$

from which we see that for ω_+ either both θ, ϕ are increasing, or they are both decreasing, but for ω_- if one angle is increasing the other is decreasing.