Math 1302, Week 7: Variable mass systems

Example: Coupling of two moving carriages

Consider two train carriages of mass m_1, m_2 moving on the same track with speeds U_1 and U_2, where $U_1 > U_2$ (see figure 1). When they catch each other up they couple together to make a single coupled pair of carriages that moves with speed V. Find V.

Solution: We have conservation of momentum:

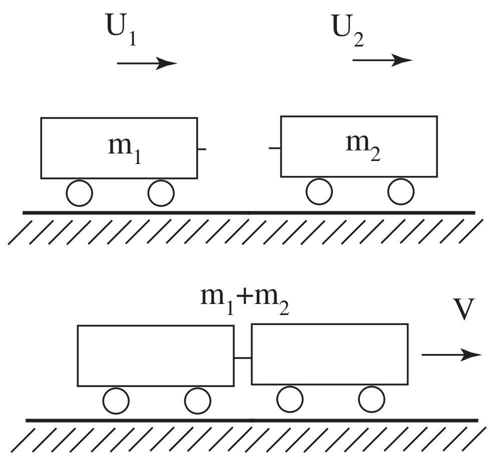

Figure 1: Two carriages coupling together into a single mass

$$\begin{aligned} \text{momentum before} &= \text{momentum after} \\ m_1U_1 + m_2U_2 &= (m_1 + m_2)V \end{aligned}$$

So that

$$V = \frac{m_1U_1 + m_2U_2}{m_1 + m_2}.$$

Thus the coupled carriages move with the speed of the centre of mass of the uncoupled carriages.

What about energy changes? Before colliding the total kinetic energy is

$$T_b = \frac{1}{2}m_1U_1^2 + \frac{1}{2}m_2U_2^2,$$

and after colliding and joining the kinetic energy is

$$T_a = \frac{1}{2}(m_1 + m_2)\left(\frac{m_1U_1 + m_2U_2}{m_1 + m_2}\right)^2$$

Hence the change in kinetic energy $\Delta T = T_a - T_b$ is given by

$$\begin{aligned}
\Delta T &= \frac{1}{2}(m_1+m_2)\left(\frac{m_1U_1+m_2U_2}{m_1+m_2}\right)^2 - \frac{1}{2}m_1U_1^2 - \frac{1}{2}m_2U_2^2 \\
&= \frac{1}{2}\frac{m_1^2U_1^2+m_2^2U_2^2+2m_1m_2U_1U_2}{m_1+m_2} - \frac{1}{2}m_1U_1^2 - \frac{1}{2}m_2U_2^2 \\
&= \frac{1}{2(m_1+m_2)}\left[m_1^2U_1^2+m_2^2U_2^2+2m_1m_2U_1U_2) - (m_1^2+m_1m_2)U_1^2 - (m_1m_2+m_2^2)U_2^2\right] \\
&= \frac{1}{2(m_1+m_2)}\left(2m_1m_2U_1U_2) - (m_1m_2)U_1^2 - (m_1m_2)U_2^2\right) \\
&= \frac{-m_1m_2(U_1-U_2)^2}{2(m_1+m_2)} < 0.
\end{aligned}$$

Thus energy is lost in the collision. (Where does it go?) Notice that this is just the same as two bodies colliding, but with coefficient of restitution $e = 0$.

Changing mass systems

When the mass of a system is changing with time, we will need a generalisation of Newton's second law. We will use that

$$\text{force} = \text{rate of change of momentum.}$$

This reduces to Newton's 2nd law when the mass is constant with time.

Motion of particle through cloud of stationary dust

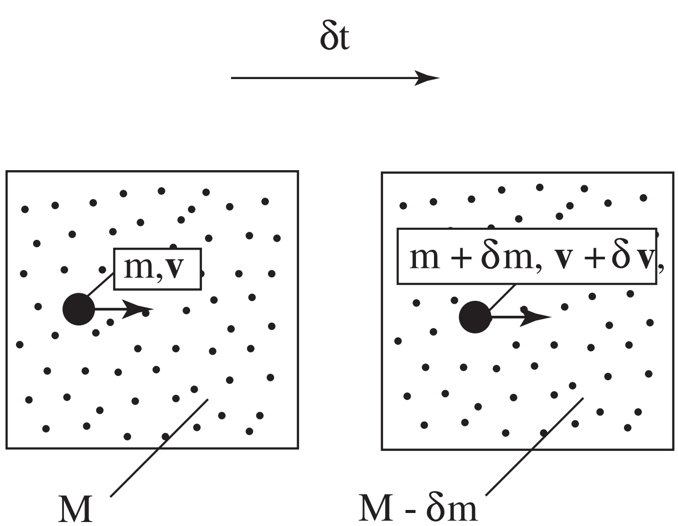

Figure 2: Particle moving through a cloud of stationary dust particles

We consider a particle moving through a cloud of stationary dust (see figure 2). The particle has mass $m(t)$ at time t, and the total mass of the dust cloud is $M(t)$. Since the cloud dust is stationary, it has no momentum. Let the velocity of the particle at time t be $\underline{v}(t)$ and at time $t + \delta t$ let it be $\underline{v} + \delta \underline{v}$. Then

$$\text{total momentum at } t = \underbrace{m\underline{v}}_{\text{momentum of particle}} + \underbrace{M\underline{0}}_{\text{momentum of dust}} = m\underline{v}.$$

$$\text{total momentum at } t+\delta t = \underbrace{(m+\delta m)(\underline{v}+\delta\underline{v})}_{\text{mtm of particle}} + \underbrace{(M-\delta m)\underline{0}}_{\text{zero mtm of dust}} = m\underline{v} + m\delta\underline{v} + \delta m\underline{v} + \delta m\delta\underline{v}.$$

Hence

$$\frac{\text{change in momentum}}{\delta t} = \frac{1}{\delta t}\left\{(m\underline{v} + m\delta\underline{v} + \delta m\underline{v} + \delta m\delta\underline{v}) - (m\underline{v})\right\}$$

Now let $\delta t \to 0$ to obtain

$$\text{rate of change of momentum} = \lim_{\delta t\to 0}\frac{1}{\delta t}(m\delta\underline{v} + \delta m\underline{v} + \delta m\delta\underline{v}) = m\frac{d\underline{v}}{dt} + \frac{dm}{dt}\underline{v} = \frac{d}{dt}(m\underline{v}).$$

When there is an external force $\underline{F}$ acting on the system we obtain

$$\underline{F} = \text{rate of change of momentum} = \frac{d}{dt}(m\underline{v}). \tag{1}$$

Example: falling raindrop

Suppose that a raindrop falls through a cloud and accumulates mass at a rate kmv where $k > 0$ is a constant, m is the mass of the raindrop, and v its velocity. What is the speed of the raindrop at a given time if it starts from rest, and what is its mass?

Solution: We are taking x as distance fallen and $v = \dot{x}$. Then the external force is its weight mg and so from (1) we have

$$mg = \frac{d}{dt}(mv) = m\frac{dv}{dt} + v\frac{dm}{dt} = m\frac{dv}{dt} + kmv^2,$$

since we are told that $dm/dt = kmv$. Cancelling the mass and rearranging

$$\frac{dv}{dt} = g - kv^2,$$

so that

$$\int_0^v \frac{dv}{g-kv^2} = \int_0^t dt = t.$$

Now set $V^2 = g/k$ and use partial fractions to get

$$t = \int_0^v \frac{dv}{g-kv^2} = \frac{1}{2kV}\int_0^v \frac{1}{V+v} + \frac{1}{V-v}\,dv = \frac{1}{2kV}\log\left(\frac{V+v}{V-v}\right)$$

so $V + v = (V - v)e^{2kVt}$, i.e. $v = V\left(\frac{e^{2kVt}-1}{e^{2kVt}+1}\right) = V\tanh(Vkt)$, so that

$$v = \sqrt{\frac{g}{k}}\tanh(\sqrt{kg}t).$$

Now we may find the mass: We have $\frac{dm}{dt} = kmv = km\sqrt{\frac{g}{k}}\tanh(\sqrt{kg}t) = m\sqrt{kg}\tanh(\sqrt{kg}t)$. Thus

$$\int_0^t \frac{1}{m}\frac{dm}{dt}\,dt = \int_0^t \sqrt{kg}\tanh(\sqrt{kg}t)\,dt$$

$$\int_{m_0}^m \frac{dm}{m} = \int_0^t \sqrt{kg}\tanh(\sqrt{kg}t)\,dt$$

$$\log m - \log m_0 = \log\cosh(\sqrt{kg}t)$$

which gives

$$m = m_0\cosh(\sqrt{kg}t).$$

Example: Raindrop falling through a cloud accumulating mass a given rate

A raindrop falls through a cloud while accumulating mass at a rate λr^2 where r is its radius (assume that the raindrop remains spherical) and $\lambda > 0$. Find its velocity at time t if it starts from rest with radius a.

Solution: We have that $\frac{dm}{dt} = \lambda r^2$. But $m = \frac{4}{3}\pi r^3\rho$ where ρ is the density. So

$$\lambda r^2 = \frac{dm}{dt} = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\rho\right) = 4\rho\pi r^2\frac{dr}{dt}.$$

This gives $dr/dt = \mu$ where $\mu = \frac{\lambda}{4\rho\pi}$, which gives

$$r = \mu t + C,$$

where C is a constant. Using $r = a$ at $t = 0$ we obtain $C = a$ and hence $r = \mu t + a$. Now we have

$$\frac{1}{m}\frac{dm}{dt} = \frac{3}{4\rho\pi r^3}\lambda r^2 = \frac{3\lambda}{4\rho\pi r} = \frac{3\mu}{(\mu t + a)}.$$

Thus from $\frac{d}{dt}(mv) = mg$ we obtain $\dot{v}m + v\dot{m} = mg$ so that $\dot{v} + \frac{\dot{m}}{m}v = g$ and hence

$$\frac{dv}{dt} + \frac{3\mu}{(\mu t + a)}v = g.$$

This can be solved using the integrating factor $I = \exp(\int \frac{3\mu}{(\mu t+a)}\,dt) = \exp(3\log(a + \mu t)) = (a + \mu t)^3$. This gives

$$\frac{d}{dt}[v(a + \mu t)^3] = g(a + \mu t)^3.$$

Integrating $v(a + \mu t)^3 = C' + \frac{g}{4\mu}(a + \mu t)^4$. Using $v(0) = 0$ we get $C' = -ga^4/4\mu$ and so

$$v(t) = \frac{g}{4\mu}\left((a + \mu t) - a^4(a + \mu t)^{-3}\right).$$

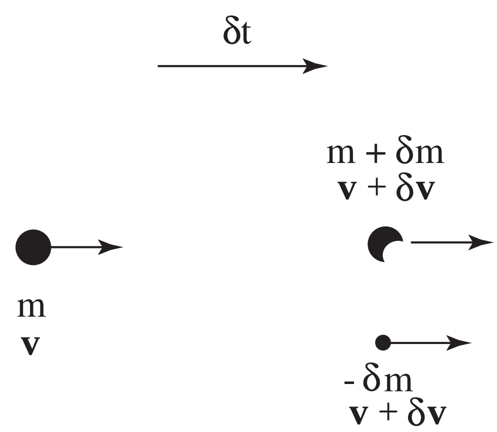

Figure 3: Particle moving and gaining $\delta m > 0$ or losing $\delta m < 0$ mass from mass moving at zero relative velocity

Mass lost or gained at zero relative velocity

The example to keep in mind here is a hot air balloon containing a bag of sand. The sand is released to control the height of the balloon. As the sand trickles out it is (effectively) stationary relative to the balloon. Consider, then, figure 3. Particle has mass m and is moving with velocity $\underline{v}$ at time t. At a later time $t + \delta t$ it has mass $m + \delta m$ and velocity $\underline{v} + \delta \underline{v}$. The mass moving with the particle with zero relative velocity has mass $-\delta m$ and velocity $\underline{v} + \delta \underline{v}$. Hence

$$\text{momentum at } t = m\underline{v},$$

and

$$\text{momentum at } t + \delta t = (m + \delta m)(\underline{v} + \delta \underline{v}) + (-\delta m)(\underline{v} + \delta \underline{v})$$

$$\begin{aligned}
\text{rate of change of momentum} &= \lim_{\delta t \to 0} \frac{1}{\delta t} \left\{ (m + \delta m)(\underline{v} + \delta \underline{v}) + (-\delta m)(\underline{v} + \delta \underline{v}) - m\underline{v} \right\} \\
&= \lim_{\delta t \to 0} \frac{1}{\delta t} \left\{ m\underline{v} + \delta m \underline{v} + m \delta \underline{v} + \delta m \delta \underline{v} - \delta m \underline{v} - \delta m \delta \underline{v} - m\underline{v} \right\} \\
&= m \frac{d\underline{v}}{dt}.
\end{aligned}$$

Hence for an external force of $\underline{F}$ we get

$$\underline{F} = m \frac{d\underline{v}}{dt}. \tag{2}$$

Balloon rising

Suppose a balloon constant mass M contains a bag of sand mass m_0 experiences a constant upward thrust of C. Initially it is in equilibrium, and then the sand is released at a constant rate so that it is all released in time t_0. Find the height of the balloon and its velocity when all the sand has been released.

Solution: As the sand leaves the bag its speed relative to the ground is the same as that of the balloon, i.e. its speed relative to the balloon is zero, so we may use (2):

$$(M+m)\frac{dv}{dt} = C - (M+m)g.$$

Thus

$$\frac{dv}{dt} = \frac{C}{M+m} - g.$$

Now the sand empties in time t_0 at a constant rate, say λ. Thus $m(t) = m_0 - \lambda t$ (solve $dm/dt = -\lambda$). If at time t_0 we have $m = 0$ then we must have $\lambda = m_0/t_0$. Hence

$$\frac{dv}{dt} = \frac{C}{M+m_0-\lambda t} - g.$$

Integrating over t:

$$v(t) - v(0) = -gt - \frac{C}{\lambda}[\log(M+m_0-\lambda t)]_0^t = -gt - \frac{C}{\lambda}\log\left(\frac{M+m_0-\lambda t}{M+m_0}\right).$$

Now if the balloon is initially in equilibrium the upward thrust equals its weight: $C = (M+m_0)g$. Thus using that $v(0) = 0$ and $\lambda = m_0/t_0$ we arrive at:

$$v(t) = -gt - \frac{(M+m_0)gt_0}{m_0}\log\left(1 - \frac{m_0 t}{(M+m_0)t_0}\right).$$

To find the height, lets first set $\alpha = m_0/(t_0(M+m_0))$. Then we have

$$\frac{dx}{dt} = v = -gt - \frac{g}{\alpha}\log(1-\alpha t).$$

Hence

$$x(t) = -\frac{gt^2}{2} - \frac{g}{\alpha}\int \log(1-\alpha t)\,dt + K.$$

We clearly need the integral $\int \log z\,dz$. To do this we can either argue that $\frac{d}{dz}(z\log z) = \log z + z/z = \log z + 1$, so that $\frac{d}{dz}(z\log z - z) = \log z$ and hence

$$\int \log z\,dz = \int \frac{d}{dz}(z\log z - z)\,dz = z\log z - z + \text{constant},$$

or we may use integration by parts after splitting $\log z = \log z \times 1$ and use $u = \log z, dv/dz = 1$, etc. The end result is that using the substitution $z = 1 - \alpha t$

$$\int \log(1-\alpha t)\,dt = \int \log z\left(\frac{-1}{\alpha}\right)dz = -\frac{1}{\alpha}z(\log z - 1) + K' = -\frac{1}{\alpha}(1-\alpha t)[\log(1-\alpha t) - 1] + K'.$$

This yields

$$x(t) = K'' - \frac{gt^2}{2} + \frac{g}{\alpha^2}(1-\alpha t)\{\log(1-\alpha t) - 1\},$$

where K'' is found from initial conditions:

$$x(t) = x(0) + \frac{g}{\alpha^2} - \frac{gt^2}{2} + \frac{g}{\alpha^2}(1-\alpha t)\{\log(1-\alpha t) - 1\}$$

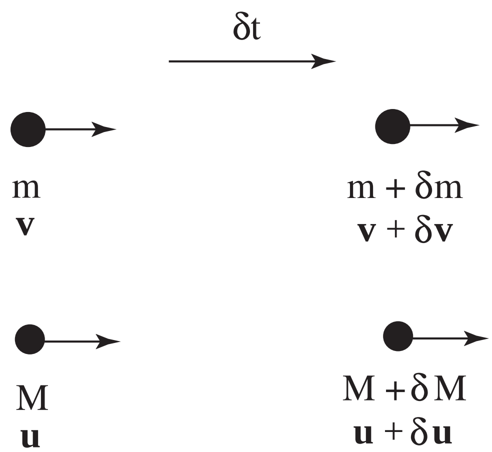

Figure 4: Particle moving with velocity $\underline{v}$ and gaining $\delta m > 0$ or losing $\delta m < 0$ mass from mass M moving at velocity $\underline{u}$.

General variable mass problems

Consider figure 4. We have

$$\text{momentum before} = m\underline{v} + M\underline{u},$$

and

$$\text{momentum after} = (m + \delta m)(\underline{v} + \delta\underline{v}) + (M + \delta M)(\underline{u} + \delta\underline{u}).$$

Thus

$$\begin{aligned}
\text{rate of change in momentum} &= \lim_{\delta t \to 0} \frac{1}{\delta t} \{(m + \delta m)(\underline{v} + \delta\underline{v}) + (M + \delta M)(\underline{u} + \delta\underline{u}) - (m\underline{v} + M\underline{u})\} \\
&= \lim_{\delta t \to 0} \frac{1}{\delta t} \{\delta m\underline{v} + m\delta\underline{v} + \delta m\delta\underline{v} + \delta M\underline{u} + M\delta\underline{u} + \delta M\delta\underline{u}\} \\
&= \underline{v}\frac{dm}{dt} + m\frac{d\underline{v}}{dt} + \underline{u}\frac{dM}{dt} + M\frac{d\underline{u}}{dt}.
\end{aligned}$$

Now, we may also use that $\delta m = -\delta M$ (mass conservation), so that $dm/dt = -dM/dt$ and

$$\boxed{\text{rate of change in momentum} = \frac{d}{dt}(m\underline{v}) - \underline{u}\frac{dm}{dt} + M\frac{d\underline{u}}{dt}.}$$

This is a general result. Note that when $\underline{u} = 0$, we obtain rate of change in momentum $= \frac{d}{dt}(m\underline{v})$ as in equation (1). Also, when $\underline{u} = \underline{v}$ we have rate of change in momentum $= (m + M)\frac{d\underline{v}}{dt}$ as in equation (2).

Example: Rocket motion (I)

A rocket of mass m emits mass backwards at speed u relative to the rocket at a constant rate k. Ignoring gravity and air resistance find its speed v at time t if at $t = 0$ it has speed v_0 and

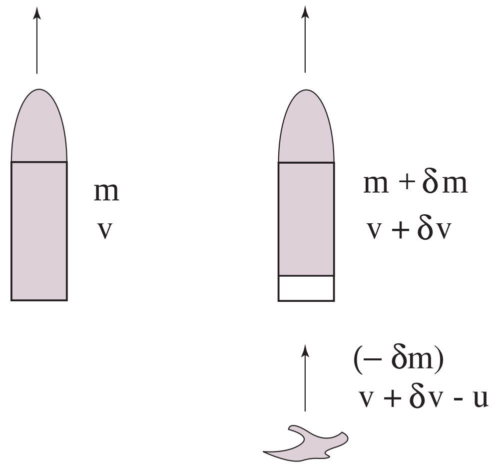

Figure 5: Rocket burning fuel that is ejected at velocity u relative to the rocket.

mass $M_0 = M + m_0$, where m_0 is the amount of fuel for burning.

Solution: Let m be the mass of the rocket and fuel at time t. We refer to figure 5.

$$\begin{aligned}\text{rate of change in momentum} &= \lim_{\delta t\to 0}\frac{1}{\delta t}\left\{(m+\delta m)(v+\delta v)+(-\delta m)(v+\delta v-u)-mv\right\}\\ &= \lim_{\delta t\to 0}\frac{1}{\delta t}\left\{\delta m v+m\delta v+\delta m\delta v-\delta m v+u\delta m-\delta m\delta v\right\}\\ &= m\frac{dv}{dt}+u\frac{dm}{dt}.\end{aligned}$$

There is no external force acting so the rate of change of momentum is zero:

$$m\frac{dv}{dt} = -u\frac{dm}{dt}.$$

But $dm/dt = -k$ is the constant loss of fuel mass. Thus $m(t) = M_0 - kt = M + m_0 - kt$ for $t < m_0/k$ (after which there is no more fuel to burn).

Thus

$$\frac{dv}{dt} = \frac{ku}{M_0 - kt},$$

which integrates to give

$$v(t) = -u\log\left(M_0 - kt\right) + K.$$

Since $t = 0$ we have $v = v_0$ we obtain

$$v(t) = v_0 - u\log\left(1 - \frac{k}{M_0}t\right), \quad (t \le m_0/k).$$

Example: Rocket motion (II)

The rocket now moves under the pull of gravity and air resistance kmv^2 and the fuel is burned so that $m = m_0 e^{-bt}$ where $b > g/u$ is a constant.

We have $F = -mg - kmv^2$ for the external force. This equals the rate of change of momentum:

$$F = m\frac{dv}{dt} + u\frac{dm}{dt}.$$

$$m\frac{dv}{dt} + u\frac{dm}{dt} = -mg - kmv^2.$$

Expanding and dividing by m,

$$\frac{dv}{dt} + \frac{u}{m}\frac{dm}{dt} = -g - kv^2.$$

Now we are told that $m = m_0 e^{-bt}$, so that $\frac{1}{m}\frac{dm}{dt} = -b$. This gives

$$\frac{dv}{dt} - bu = -g - kv^2,$$

which we rewrite as

$$\frac{dv}{dt} = \lambda^2 - kv^2, \text{ where } \lambda^2 = bu - g > 0.$$

We use partial fractions to obtain

$$\int \frac{1}{\lambda - \sqrt{k}v} + \frac{1}{\lambda + \sqrt{k}v}\, dv = 2\lambda t + C,$$

where C is a constant. Thus

$$\log\left(\frac{\lambda + \sqrt{k}v}{\lambda - \sqrt{k}v}\right) = 2\lambda\sqrt{k}t + C'.$$

Using that $v(0) = 0$ we get $C' = 0$. Tidying up we finally obtain

$$v(t) = \frac{\lambda}{\sqrt{k}}\tanh(\lambda\sqrt{k}t).$$

As $t \to \infty$ we have

$$\begin{aligned} v \quad &\to \quad \lim_{t\to\infty} \frac{\lambda}{\sqrt{k}}\tanh(\lambda\sqrt{k}t) \\ &= \quad \lim_{t\to\infty} \frac{\lambda}{\sqrt{k}}\left(\frac{1 - e^{-2\lambda\sqrt{k}t}}{1 + e^{-2\lambda\sqrt{k}t}}\right) = \frac{\lambda}{\sqrt{k}}. \end{aligned}$$

Thus the rocket reaches a limiting velocity $\frac{\lambda}{\sqrt{k}} = \sqrt{\frac{bu-g}{k}}$.

Example (2004 Exam question 5)

The mass of a spacecraft at time t is $m(t)$ and its velocity is $\underline{V}(t)$. For $t < 0$, $m(t) = M$ and $\underline{V} = U\underline{i}$ where M and U are constants and $\underline{i}$ is a constant unit vector. For $0 < t < T$, the craft encounters a stream of particles which have velocity $w(\cos\alpha\underline{i} + \sin\alpha\underline{j})$ where w and α are constants and $\underline{j}$ is a unit vector orthogonal to $\underline{i}$.

A constant mass ρ of the particles enter the craft per unit time and these particles are thereafter stationary relative to the craft.

(a) If $\underline{V} = u\underline{i} + v\underline{j}$, show that

$$m\frac{du}{dt} + \frac{dm}{dt}u = \rho w \cos\alpha, \quad m\frac{dv}{dt} + \frac{dm}{dt}v = \rho w \sin\alpha, \quad \frac{dm}{dt} = \rho.$$

(b) Solve these equations to show that at time T the direction of motion of the craft has been turned through an angle β, where

$$\tan\beta = \frac{\rho w T \sin\alpha}{MU + \rho w T \cos\alpha}$$

Solution: Consider the before and after picture shown in figure 6 for the momentum change

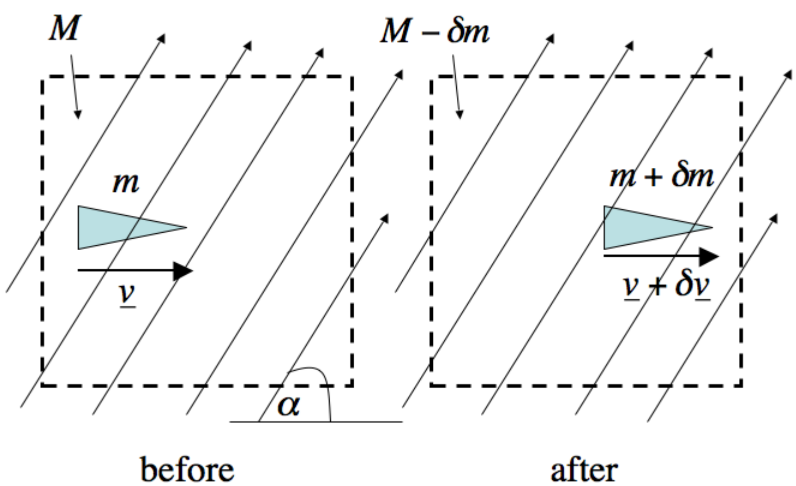

Figure 6: Rocket in a stream of particles

in time δt. Since the particles pass through the box at a constant rate, in the absence of the spaceship the mass inside a fixed virtual box (we need only consider 2D) is some constant M. Suppose the spaceship is passing through the box and is there at t (before figure) and $t + \delta t$ (after figure). At time t the momentum inside the box is

$$M(w\cos\alpha\underline{i} + w\sin\alpha\underline{j}) + m\underline{v},$$

whereas at time $t + \delta t$ it is

$$(M - \delta m)(w\cos\alpha\underline{i} + w\sin\alpha\underline{j}) + (m + \delta m)(\underline{v} + \delta\underline{v}).$$

Since there is no external force, these must be the same:

$$M(w\cos\alpha\underline{i} + w\sin\alpha\underline{j}) + m\underline{v} = (M - \delta m)(w\cos\alpha\underline{i} + w\sin\alpha\underline{j}) + (m + \delta m)(\underline{v} + \delta\underline{v})$$

Thus expanding and cancelling we obtain

$$m\delta\underline{v} + \delta m\underline{v} + \delta m\delta\underline{v} - \delta m(w\cos\alpha\underline{i} + w\sin\alpha\underline{j}) = \underline{0}.$$

Now divide by δt and take the limit as δt tends to zero:

$$m\dot{\underline{v}} + \dot{m}\underline{v} = \dot{m}(w\cos\alpha\underline{i} + w\sin\alpha\underline{j}).$$

Mass is accumulating in the spacecraft at rate ρ so that $dm/dt = \rho$. This gives

$$m\dot{u} + \rho u = \rho w\cos\alpha \qquad (3)$$

$$m\dot{v} + \rho v = \rho w\sin\alpha \qquad (4)$$

which is the same as the equations stated (since $\dot{m} = \rho$).

Now $\dot{m} = \rho$ can be integrated to give $m = C + \rho t$ where C is constant. But at $t = 0$, $m(0) = M$ so that $C = M$ and $m = M + \rho t$. From (4) we get

$$(M + \rho t)\dot{u} + \rho u = \rho w\cos\alpha.$$

The LHS is just $\frac{d}{dt}[(M + \rho t)u]$ and hence

$$\frac{d}{dt}[(M + \rho t)u] = \rho w\cos\alpha,$$

which integrates to

$$(M + \rho t)u = K + \rho w\cos\alpha t,$$

where K is a constant. At $t = 0$, $u = U$, and so $K = MU$ and hence

$$u(t) = \frac{MU + (\rho w\cos\alpha)t}{M + \rho t}.$$

Similarly,

$$(M + \rho t)v = K' + \rho w\sin\alpha t,$$

where K' is a constant. But $v(0) = 0$ and hence $K' = 0$.

After time T, the spaceship has turned through angle β where $\tan\beta = \frac{v(T)}{u(T)}$ which gives

$$\tan\beta = \frac{(\rho w\sin\alpha)T}{MU + (\rho w\cos\alpha)T}.$$