Math 1302, Week 6: Interacting particle systems

Relative velocity

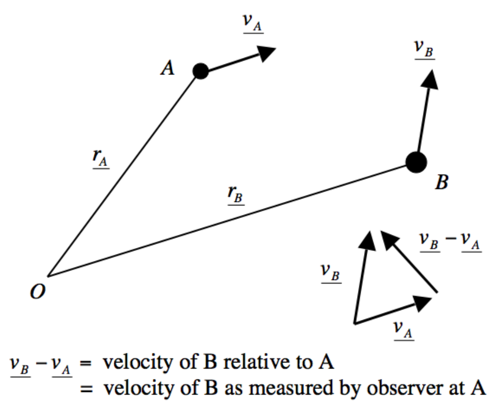

Figure 1: Relative velocity of two moving bodies

Consider two particles A, B moving with velocities $\underline{v_A}$ and $\underline{v_B}$ relative to a fixed origin O. Then the velocity of A relative to B is the velocity of A as measured by an observer moving with B.

$$\text{velocity of A relative to B} = \underline{v_A} - \underline{v_B}.$$

Similarly

$$\text{velocity of B relative to A} = \underline{v_B} - \underline{v_A}.$$

Consider a boat crossing a flowing river. Suppose the river flows with velocity $\underline{v_{RG}}$ relative to

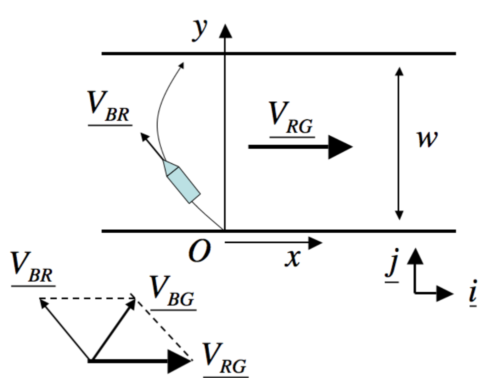

Figure 2: Boat crossing a river

the stationary bank. Suppose also that the boat moves with velocity $\underline{v_{BR}}$ relative to the river (so it moves with velocity $\underline{v_{BR}}$ in a still lake). Then

$$\underline{v_{BR}} = \underline{v_{BG}} - \underline{v_{RG}}$$

Thus rearranging

$$\underline{v_{BG}} = \underline{v_{BR}} + \underline{v_{RG}}.$$

We may use $\underline{v_{BG}}$ to calculate where the boat is at a given time. If the flow of the river is a constant then

$$\underline{r}(t) = x(t)\underline{i} + y(t)\underline{j} = \underline{v_{BG}}t = \underline{v_{BR}}t + \underline{v_{RG}}t.$$

Thus to find where the boat ends up, just find t^* such that $y(t^*) = w$ and then $x(t^*) = \underline{i} \cdot (\underline{v_{BR}} + \underline{v_{RG}})t^*$.

Mutual attraction between particles

Consider two particles masses m_1, m_2 interacting under a mutual attractive force (see figure 3). Thus m_1 experiences a force $\underline{F}_1$ due to m_2, and m_2 experiences a force $\underline{F}_2$ due to m_1. By

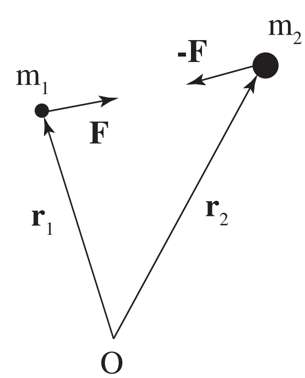

Figure 3: Two particles interacting through mutual attractive force

Newton's 3rd law $\underline{F}_2 = -\underline{F}_1$. Let $\underline{F}_1 = \underline{F}$ so that $\underline{F}_2 = -\underline{F}$. The dynamical equations are from Newton's 2nd law:

$$m_1\ddot{\underline{r}}_1 = \underline{F} \text{ and } m_2\ddot{\underline{r}}_2 = -\underline{F}.$$

Adding these two equations we obtain $m_1\ddot{\underline{r}}_1 + m_2\ddot{\underline{r}}_2 = \underline{F} - \underline{F} = \underline{0}$. We may take the second derivative operator outside to give

$$\frac{d^2}{dt^2}(m_1\underline{r}_1 + m_2\underline{r}_2) = \underline{0}.$$

Now divide both sides by $m_1 + m_2$ to obtain

$$\frac{d^2}{dt^2}\left(\frac{m_1\underline{r}_1 + m_2\underline{r}_2}{m_1 + m_2}\right) = \underline{0}.$$

We now define

$$\boxed{\underline{r}_C = \frac{m_1\underline{r}_1 + m_2\underline{r}_2}{m_1 + m_2} = \text{the centre of mass of } m_1, m_2}$$

so that $\ddot{\underline{r}}_C = \underline{0}$. This says that the centre of mass of the two particles under a mutually

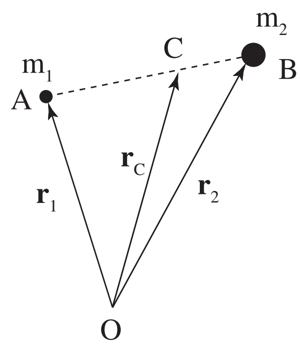

Figure 4: Location of the centre of mass C of two particles A, B. We have $\dfrac{|AC|}{|AB|} = \dfrac{m_2}{m_1+m_2}$ and $\dfrac{|CB|}{|AB|} = \dfrac{m_1}{m_1+m_2}$. Thus the centre of mass is situated towards the more massive particle along the line joining them.

attractive force moves with zero acceleration. Put another way, the centre of mass of the two particles moves with constant velocity.

For example, if two particles of equal mass move along a line towards each other under a mutually attractive force when starting from rest, then the centre of mass remains stationary and at the halfway point between the particles.

More generally, when the forces are not necessarily mutually attractive we have

$$m_1\ddot{\underline{r}}_1 = \underline{F}_1 \quad \text{and} \quad m_2\ddot{\underline{r}}_2 = \underline{F}_2.$$

Adding these equations we obtain

$$m_1\ddot{\underline{r}}_1 + m_2\ddot{\underline{r}}_2 = \underline{F}_1 + \underline{F}_2$$

We may rewrite this as

$$(m_1+m_2)\left(\frac{m_1\ddot{\underline{r}}_1 + m_2\ddot{\underline{r}}_2}{m_1+m_2}\right) = \underline{F}_1 + \underline{F}_2$$

That is $M\ddot{\underline{r}}_C = \underline{F}$ where $M = m_1 + m_2$ is the total mass and $\underline{F} = \underline{F}_1 + \underline{F}_2$ is the total force. This shows that $\underline{r}_C$ moves like a particle of mass $M = m_1 + m_2$ under the force $\underline{F} = \underline{F}_1 + \underline{F}_2$.

These ideas can be extended for systems consisting of k particles of masses $m_1, m_2, \ldots, m_k$ positioned at $\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_k$. We define

$$\boxed{\underline{r}_C = \frac{\sum_{i=1}^k m_i\underline{r}_i}{\sum_{i=1}^k m_i} = \text{the centre of mass of } m_1, m_2, \ldots, m_k,}$$

If the ith particle is subject to the force $\underline{F}_i$ then

$$\left(\sum_{j=1}^k m_j\right)\frac{d^2}{dt^2}\underline{r}_C = \sum_{j=1}^k \underline{F}_j.$$

This shows that $\underline{r}_C$ moves like a particle of mass $M = \sum_{j=1}^k m_j$ under the force $\underline{F} = \sum_{j=1}^k \underline{F}_j$.

Angular momentum of a particle system

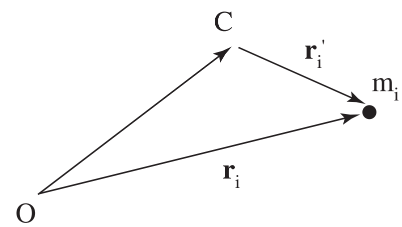

Figure 5: $\underline{r}_i'$ is the position of particle mass m_i relative to the centre of mass C.

Recall that the angular momentum about the origin O of a particle mass m at position $\underline{r}$ and moving with velocity $\underline{v} = \dot{\underline{r}}$ is $\underline{h}$ where

$$\boxed{\underline{h} = m\underline{r} \times \underline{v} \ \text{ angular momentum.}}$$

Now consider a system of particles masses $m_1, m_2, \ldots, m_k$, positions $\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_k$ and velocities $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k$. We are interested in expressing the angular momentum $\underline{h}_C$ of the system about the centre of mass C in terms of the angular momentum $\underline{h}_O$ of the system about the origin O. We have $\underline{r}_i' = \underline{r}_i - \underline{r}_C$ and $\underline{v}_i' = \underline{v}_i - \underline{v}_C$, where $\underline{v}_i = \dot{\underline{r}}_i$ and $\underline{v}_C$ is the velocity of the centre of mass (see figure 5). We compute

$$\begin{aligned} m_i(\underline{r}_i' \times \underline{v}_i') &= m_i(\underline{r}_i - \underline{r}_C) \times (\underline{v}_i - \underline{v}_C) \\ &= m_i[\underline{r}_i \times \underline{v}_i - \underline{r}_C \times \underline{v}_i - \underline{r}_i \times \underline{v}_C + \underline{r}_C \times \underline{v}_C] \end{aligned}$$

Summing over i we obtain

$$\begin{aligned} \underline{h}_C &= \sum_{i=1}^k m_i(\underline{r}_i' \times \underline{v}_i') \\ &= \sum_{i=1}^k m_i[\underline{r}_i \times \underline{v}_i - \underline{r}_C \times \underline{v}_i - \underline{r}_i \times \underline{v}_C + \underline{r}_C \times \underline{v}_C] \\ &= \sum_{i=1}^k m_i\underline{r}_i \times \underline{v}_i - \underline{r}_C \times \sum_{i=1}^k m_i\underline{v}_i - \sum_{i=1}^k m_i\underline{r}_i \times \underline{v}_C + \left(\sum_{i=1}^k m_i\right)\underline{r}_C \times \underline{v}_C \\ &= \sum_{i=1}^k m_i\underline{r}_i \times \underline{v}_i - \left(\sum_{i=1}^k m_i\right)\underline{r}_C \times \left(\frac{\sum_{i=1}^k m_i\underline{v}_i}{\left(\sum_{i=1}^k m_i\right)}\right) \\ &\quad - \left(\sum_{i=1}^k m_i\right)\left(\frac{\sum_{i=1}^k m_i\underline{r}_i}{\left(\sum_{i=1}^k m_i\right)}\right) \times \underline{v}_C + \left(\sum_{i=1}^k m_i\right)\underline{r}_C \times \underline{v}_C \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^{k} m_i \underline{r}_i \times \underline{v}_i - M\underline{r}_C \times \underline{v}_C - M\underline{r}_C \times \underline{v}_C + M\underline{r}_C \times \underline{v}_C \\
&= \sum_{i=1}^{k} m_i \underline{r}_i \times \underline{v}_i - M\underline{r}_C \times \underline{v}_C \\
&= \underline{h}_O - M\underline{r}_C \times \underline{v}_C
\end{aligned}$$

where $M = \sum_{i=1}^{k} m_i$ is the total mass of the system. Hence we have shown

$$\boxed{\underbrace{\underline{h}_O}_{\text{angular mtm about O}} = \underbrace{\underline{h}_C}_{\text{angular mtm about C}} + \underbrace{M\underline{r}_C \times \underline{v}_C}_{\text{angular mtm of centre of mass}}}$$

Example

Two bodies A, B move under gravitational attraction. Show that the angular momentum of B relative to A is constant.
Solution: Let m_1, m_2 be the masses of A, B at positions $\underline{r}_1, \underline{r}_2$ respectively. Then $m_1\ddot{\underline{r}}_1 = \underline{F}$ and $m_2\ddot{\underline{r}}_2 = -\underline{F}$. The angular momentum of B relative to A is

$$\underline{h} = (\underline{r}_2 - \underline{r}_1) \times m_2 \frac{d}{dt}(\underline{r}_2 - \underline{r}_1).$$

Thus

$$\begin{aligned}
\frac{d}{dt}\underline{h} &= \frac{d}{dt}(\underline{r}_2 - \underline{r}_1) \times m_2 \frac{d}{dt}(\underline{r}_2 - \underline{r}_1) + (\underline{r}_2 - \underline{r}_1) \times m_2 \frac{d^2}{dt^2}(\underline{r}_2 - \underline{r}_1) \\
&= \underline{0} + (\underline{r}_2 - \underline{r}_1) \times m_2(\ddot{\underline{r}}_2 - \ddot{\underline{r}}_1) = -m_2(\underline{r}_2 - \underline{r}_1) \times (\frac{\underline{F}}{m_2} + \frac{\underline{F}}{m_1}) \\
&= -m_2 \left(\frac{1}{m_1} + \frac{1}{m_2} \right) (\underline{r}_2 - \underline{r}_1) \times \underline{F}
\end{aligned}$$

Now the gravitational force $\underline{F} = Gm_1m_2(\underline{r}_2 - \underline{r}_1)|\underline{r}_2 - \underline{r}_1|^{-3}$, so that

$$\frac{d}{dt}\underline{h} = -m_2 \left(\frac{1}{m_1} + \frac{1}{m_2} \right) (\underline{r}_2 - \underline{r}_1) \times Gm_1m_2(\underline{r}_2 - \underline{r}_1)|\underline{r}_2 - \underline{r}_1|^{-3} = \underline{0},$$

since $(\underline{r}_2 - \underline{r}_1) \times (\underline{r}_2 - \underline{r}_1) = \underline{0}$.

Collision problems

Consider 2 smooth billiard balls A, B moving in a line on a smooth table, so that they collide along their line of centres (see figure 6). Suppose that before collision ball A of mass m_1 is moving with speed U_1 and ball B of mass m_2 is moving with speed U_2 where $U_1 > U_2$ (so that A catches B up). We would like to know the speed V_1 of A and the speed V_2 of B after the collision. We claim that linear momentum is conserved during the collision. Let's now sketch a proof of this.

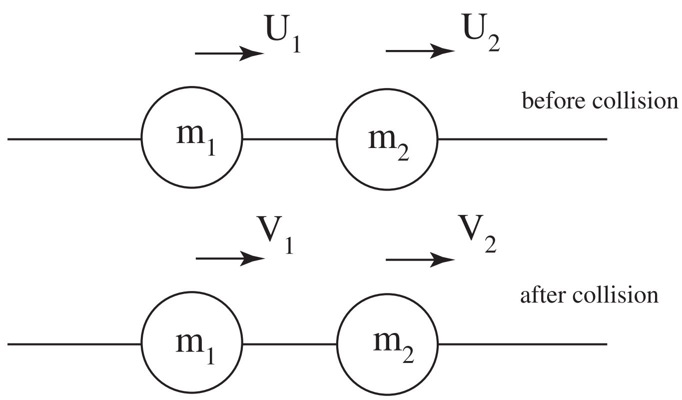

Figure 6: 2 spheres colliding along a line through their centres.

Conservation of momentum during collisions

Suppose that the spheres where able to deform, so that the collision is not instantaneous but occurs over some small time period δt (see figure 7). During the collision Newton's 3rd law

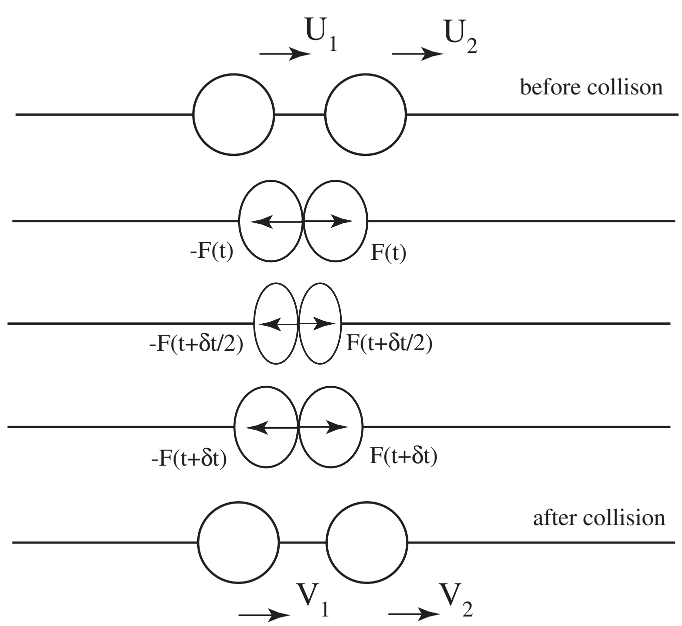

Figure 7: 2 spheres colliding along a line through their centres.

shows that each sphere exerts an equal and opposite force on the other. Thus we have

$$\begin{aligned} m_1\ddot{\underline{r}}_1 = \underline{0}, \;\; m_2\ddot{\underline{r}}_2 = \underline{0} &\quad \text{before collision} \\ m_1\ddot{\underline{r}}_1 = \underline{F}(t), \;\; m_2\ddot{\underline{r}}_2 = -\underline{F}(t) &\quad \text{during collision} \\ m_1\ddot{\underline{r}}_1 = \underline{0}, \;\; m_2\ddot{\underline{r}}_2 = \underline{0} &\quad \text{after collision} \end{aligned}$$

Thus $m_1\ddot{\underline{r}}_1 + m_2\ddot{\underline{r}}_2 = \underline{0}$ for all t (the forces always sum to zero). Thus integrating $m_1\dot{\underline{r}}_1 + m_2\dot{\underline{r}}_2 =$ constant, i.e. linear momentum is conserved during the collision. This is true for all collision periods δt, so we conclude that even when the spheres do not deform, momentum is still conserved during the collision.

Hence in the collision of figure 6 we have from conservation of momentum:

$$m_1U_1 + m_2U_2 = m_1V_1 + m_2V_2.$$

We seek the post-collision velocities, so we need another equation. This comes from the combined properties of the spheres: Empirically one finds that

$$e := -\frac{\text{relative velocity of separation of A,B}}{\text{relative velocity of approach of A,B}}$$

is a constant for collisions between two given spheres. Thus we have

$$e = \frac{V_2 - V_1}{U_1 - U_2}.$$

This is a non-negative number since $U_1 > U_2$ (for A to catch B up) and after the collision B moves away from A, so $V_2 > V_1$. Thus to find V_1, V_2 we must solve

$$\begin{aligned} m_1U_1 + m_2U_2 &= m_1V_1 + m_2V_2 \quad \text{momentum conservation} \\ e(U_1 - U_2) &= V_2 - V_1 \quad \text{restitution} \end{aligned}$$

One may write this as

$$\begin{pmatrix} m_1 & m_2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} m_1U_1 + m_2U_2 \\ e(U_1 - U_2) \end{pmatrix},$$

that is

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \frac{1}{m_1 + m_2} \begin{pmatrix} 1 & -m_2 \\ 1 & m_1 \end{pmatrix} \begin{pmatrix} m_1U_1 + m_2U_2 \\ e(U_1 - U_2) \end{pmatrix},$$

giving

$$\boxed{\begin{aligned} V_1 &= \frac{(m_1 - em_2)U_1 + m_2(1+e)U_2}{m_1 + m_2} \\ V_2 &= \frac{m_1(1+e)U_1 + (m_2 - em_1)U_2}{m_1 + m_2} \end{aligned}} \tag{1}$$

Loss of energy during collision

Before the collision, the kinetic energy is $T_b = \frac{m_1}{2}U_1^2 + \frac{m_2}{2}U_2^2$ and after it is $T_a = \frac{m_1}{2}V_1^2 + \frac{m_2}{2}V_2^2$. Thus the energy change from the collision is

$$\Delta T = T_a - T_b = \frac{m_1}{2}V_1^2 + \frac{m_2}{2}V_2^2 - \frac{m_1}{2}U_1^2 - \frac{m_2}{2}U_2^2.$$

Note that (1) can be rearranged to give

$$V_1 = U_1 - \frac{m_2(1+e)}{M}\Delta U, \quad V_2 = U_2 + \frac{m_1(1+e)}{M}\Delta U,$$

where $M = m_1 + m_2$ and $\Delta U = U_1 - U_2$. Thus

$$V_1 - U_1 = -\frac{m_2(1+e)}{M}\Delta U, \quad U_1 + V_1 = 2U_1 - \frac{m_2(1+e)}{M}\Delta U,$$

$$V_2 - U_2 = \frac{m_1(1+e)}{M}\Delta U, \quad U_2 + V_2 = 2U_2 + \frac{m_1(1+e)}{M}\Delta U.$$

This gives

$$\begin{aligned}
\Delta T &= \frac{m_1}{2}V_1^2 + \frac{m_2}{2}V_2^2 - \frac{m_1}{2}U_1^2 - \frac{m_2}{2}U_2^2 \\
&= \frac{m_1}{2}(V_1^2 - U_1^2) + \frac{m_2}{2}(V_2^2 - U_2^2) \\
&= \frac{m_1}{2}(V_1 + U_1)(V_1 - U_1) + \frac{m_2}{2}(V_2 + U_2)(V_2 - U_2) \\
&= \frac{m_1}{2}\left[\left(2U_1 - \frac{m_2(1+e)\Delta U}{M}\right)\left(-\frac{m_2(1+e)}{M}\Delta U\right)\right] \\
&\quad + \frac{m_2}{2}\left[\left(2U_2 + \frac{m_1(1+e)\Delta U}{M}\right)\left(\frac{m_1(1+e)}{M}\Delta U\right)\right] \\
&= \frac{m_1 m_2(1+e)\Delta U}{2M}\left\{-2U_1 + \frac{m_2(1+e)\Delta U}{M} + 2U_2 + \frac{m_1(1+e)\Delta U}{M}\right\} \\
&= \frac{m_1 m_2(1+e)\Delta U}{2M}(-2\Delta U + (1+e)\Delta U)
\end{aligned}$$

So that

$$\boxed{\Delta T = \frac{m_1 m_2(e^2-1)(U_1-U_2)^2}{2M}}$$

Thus we see that if $e \in (0,1)$ energy is lost in the collision, as $e^2 < 1$ and $\Delta T < 0$. If $e = 1$ we see that no energy is lost, since $\Delta T = 0$ and then the collision is said to be *perfectly elastic*. When $e = 0$ the collision is said to be *perfectly inelastic*, and in this case $V_1 = V_2$, so the two spheres move off together at the same speed.

$e = 0 \Rightarrow$	$\Delta T < 0$	perfectly inelastic
$e \in (0,1) \Rightarrow$	$\Delta T < 0$	energy lost, inelastic
$e = 1 \Rightarrow$	$\Delta T = 0$	perfectly elastic.

Example: 3 balls in a line

Suppose there are 3 balls masses m_1, m_2, m_3 in a line (in that order) and initially they are at rest. The first ball of mass m_1 is then given a nudge so that it collides with the mass m_2 with speed U. If the coefficient of restitution of each of the balls is e, find the final speed of the ball mass m_3.
Solution: First collision: Let the ball speeds before the first collision be $U_1, U_2(=0), U_3(=0)$ and after the first collision be $V_1, V_2, V_3(=0)$. Then from (1) we have

$$V_2 = \frac{m_1(1+e)U_1 + (m_2 - em_1)U_2}{m_1 + m_2} = \frac{m_1(1+e)U}{m_1 + m_2}$$

Thus the second ball hits the stationary third ball with V_2 as given by this expression. To find the velocity of mass m_3 after the second collision we use the same formula, but replace m_1 by m_2, m_2 by m_3 and U by V_2. Thus the final speed of the third ball is

$$\frac{m_2(1+e)V_2}{m_2+m_3} = \frac{m_2(1+e)}{m_2+m_3} \times \frac{m_1(1+e)U}{m_1+m_2} = \frac{m_1 m_2 (1+e)^2 U}{(m_1+m_2)(m_2+m_3)}.$$

Example: bouncing ball

A 1kg ball falls under gravity and bounces on a table. If it hits the table the first time with speed U, show that its speed after the kth bounce is $e^k U$.
Solution: Let M be the mass of the earth (i.e. about 10^{24}kg). We have conservation of momen-

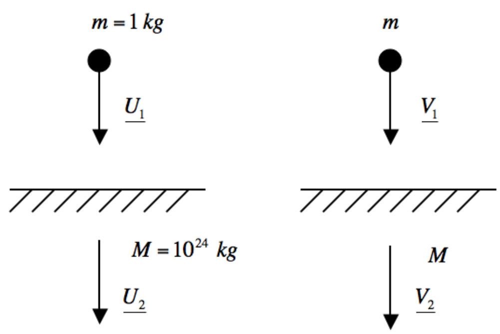

Figure 8: Ball bouncing off the Earth. The Earth gains momentum MV_2 from the collision, but its mass M is so large ($\approx 10^{24}$kg) that the Earth's change in velocity V_2 is indistinguishable from zero.

tum:

$$mU_1 + 0M = m(V_1) + MV_2$$

and

$$V_2 - V_1 = eU_1.$$

Thus $V_2 = \frac{m}{M}(U_1 - V_1)$ and $V_1 = V_2 - eU_1 = \frac{m}{M}(U_1 - V_1) - eU_1$. Rearranging and using $U_1 = U$ gives

$$V_1 = \left(\frac{\frac{m}{M} - e}{1 + \frac{m}{M}} \right) U.$$

Now $m/M = 10^{-24}$, so that to an excellent approximation, the velocity of the ball after the first bounce is

$$V_1 = -eU,$$

so that its speed is eU. On the second bounce it is $e(eU) = e^2 U$ and so on with the speed on the kth bounce being $e^k U$. Notice that the Earth gains momentum $MV_2 = m(1-e)U$ from the collision, which is not small (compared with m/M). However, $V_2 = \frac{m}{M}(1-e)U$ is small. Thus while the Earth gains momentum from the collision it gains negligible speed since it is so massive.

Example: A simple model for Newton's cradle[1]

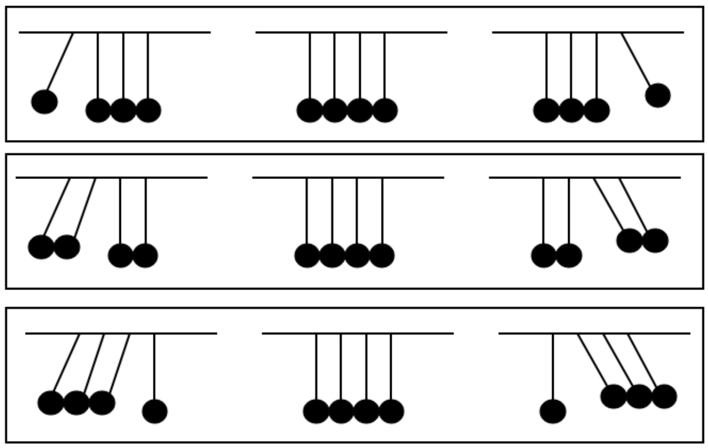

Figure 9: The motion for a 4-ball Newton's cradle. Top: one ball displaced results in last ball only moving. Middle: Two balls displaced leads to only last two balls moving. Three balls displaced leads to only last three balls moving.

The idea of Newton's cradle is that the balls, being steel, provide "perfectly elastic" collisions. All the balls are identical with mass m, say $m = 1$ for simplicity. As the balls hang vertically, and so that they just "barely" touch, we may think of the motion as being a series of collisions of identical balls along their line of centres: The middle picture of the figure shows their configuration at the instant of a collision - just as if they were moving in a line.

Thus we just work with the same problem of 4 identical balls moving in a line. First consider the top motion (1 ball moving, 3 stationary). The collisions are all elastic so when any ball moving with speed U impacts on a ball at rest then the incoming ball stops dead and the outgoing ball moves with speed U (to see this, just set $m_1 = m_2$ and $e = 1$ in equations (1)).

Thus we have the sequence of collisions shown in figure 10. Each collision stops one ball dead and sends the second ball off at the same speed, so the energy propagates through the line of balls. At each time instant only one ball moves. The final outcome is that the last ball moves off with speed U, with the previous 3 stationary. In the actual cradle the last mass then moves upwards in pendulum-like motion and the returns the last ball to collide in the opposite direction with speed U and the process is reversed. For 2 balls moving, and 2 stationary, the same principles

[1]There are different explanations around, so this is just one possible explanation for the motion of Newton's cradle.

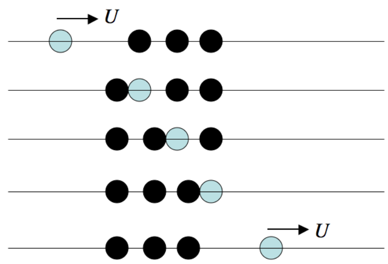

Figure 10: Model for 4-ball Newton's cradle: 1 ball incoming, 3 stationary. The balls can be made to rest arbitrarily close. A stationary ball is black, and a ball moving with speed is shaded grey. The oscillation of the cradle is modelled by sending the balls back with speed U, and so on.

work, but this time at each time instant two balls are moving, so a 2-ball 'pulse' propagates through the line (see figure 11). The remaining case, namely 3 balls incoming, 1 stationary is

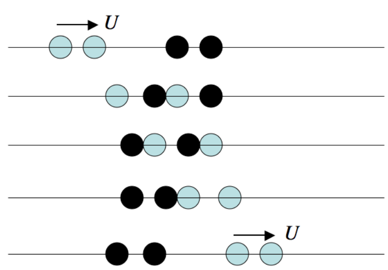

Figure 11: Model for 4-ball Newton's cradle: 2 balls incoming, 2 stationary.

shown in figure 12.

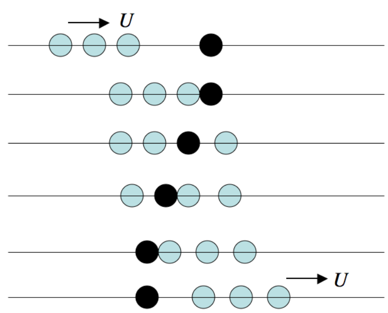

Figure 12: Model for 4-ball Newton's cradle: 3 balls incoming, 1 stationary.