Math 1302, Week 4: Motion on a surface in 3D

While motion is in 3D, the surface is two dimensional, so the motion is only really two dimensional. We will be concerned with motion on a surface of revolution. For such motion, the best coordinates to choose, since these are axisymmetric problems, are a three-dimensional extension of plane polar coordinates called *cylindrical polar coordinates*.

Cylindrical polar coordinates

Consider figure 1. The cylindrical polar coordinates are ρ, θ, z. The position vector $\underline{r}$ of the point P is given by $\underline{r} = \rho\hat{\underline{r}} + z\underline{k}$. Here

$$\rho = \sqrt{x^2 + y^2} \quad \text{and} \quad |\underline{r}| = r = \sqrt{x^2 + y^2 + z^2} = \sqrt{\rho^2 + z^2}.$$

The conversions between cylindrical polar coordinates and cartesian coordinates are

$$x = \rho\cos\theta,\ y = \rho\sin\theta, z = z \quad \leftrightarrow \quad \rho = \sqrt{x^2 + y^2}, \theta = \tan^{-1}\frac{y}{x}, z = z.$$

We will be using this form of $\underline{r}$ to examine the motion of particles on surfaces and hence we'll

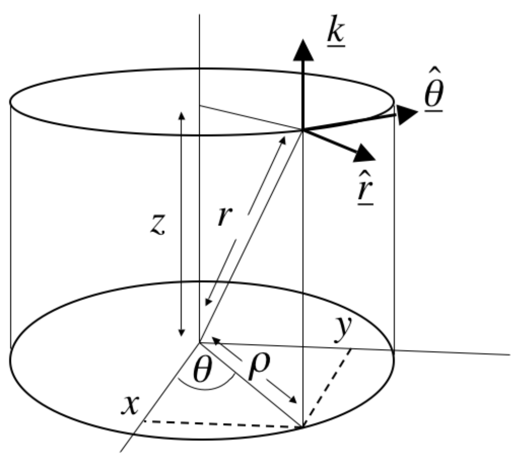

Figure 1: Cylindrical polar coordinates

need the velocity $\dot{\underline{r}}$ and the acceleration $\ddot{\underline{r}}$ in these new coordinates. We have

$$\dot{\underline{r}} = \frac{d}{dt}(\rho\hat{\underline{r}} + z\underline{k}) = \frac{d}{dt}(\rho\hat{\underline{r}}) + \dot{z}\underline{k} = \dot{\rho}\hat{\underline{r}} + \rho\dot{\theta}\hat{\underline{\theta}} + \dot{z}\underline{k},$$

where we have simply taken the expression for $\frac{d}{dt}(\rho\hat{\underline{r}})$ from the equation for velocity in plane polar coordinates (simply replace the r in $\underline{v} = r\hat{\underline{r}}$ by ρ), since in plane polars $\rho\hat{\underline{r}}$ is just the position vector of a point. For the acceleration

$$\ddot{\underline{r}} = \frac{d^2}{dt^2}(\rho\hat{\underline{r}}) + \ddot{z}\underline{k} = (\ddot{\rho} - \rho\dot{\theta}^2)\hat{\underline{r}} + \frac{1}{\rho}\frac{d}{dt}(\rho^2\dot{\theta})\hat{\underline{\theta}} + \ddot{z}\underline{k},$$

where we have used the formula for $\frac{d^2}{dt^2}(\rho\hat{\underline{r}})$ from plane polars. Thus

$$
\begin{aligned}
\underline{v} &= \dot{\rho}\underline{\hat{r}} + \rho\dot{\theta}\underline{\hat{\theta}} + \dot{z}\underline{k} \\
\underline{a} &= (\ddot{\rho} - \rho\dot{\theta}^2)\underline{\hat{r}} + \frac{1}{\rho}\frac{d}{dt}(\rho^2\dot{\theta})\underline{\hat{\theta}} + \ddot{z}\underline{k}.
\end{aligned}
$$

Example: motion on inner surface of cylinder

Let $\underline{F}$ be the total force acting on the particle mass m: then $\underline{F} = -mg\underline{k} + \underline{R}$ where $\underline{R}$ is the reaction force of the cylinder wall on the particle. We take $\underline{R} = -R\underline{\hat{r}}$, since the reaction will be in the opposite direction of the unit vector $\underline{\hat{r}}$ and then $R \geq 0$. Thus from $m\underline{\ddot{r}} = \underline{F}$ we obtain

$$
m\left\{(\ddot{\rho} - \rho\dot{\theta}^2)\underline{\hat{r}} + \frac{1}{\rho}\frac{d}{dt}(\rho^2\dot{\theta})\underline{\hat{\theta}} + \ddot{z}\underline{k}\right\} = -R\underline{\hat{r}} - mg\underline{k}.
$$

Equating components of $\underline{\hat{r}}, \underline{\hat{\theta}}, \underline{k}$ (since they are mutually orthogonal) we obtain

$$
\begin{aligned}
m(\ddot{\rho} - \rho\dot{\theta}^2) &= -R && (1) \\
\frac{m}{\rho}\frac{d}{dt}(\rho^2\dot{\theta}) &= 0 && (2) \\
m\ddot{z} &= -mg. && (3)
\end{aligned}
$$

From (2) we see that $\frac{d}{dt}(\rho^2\dot{\theta}) = 0$, so that $\rho^2\dot{\theta} = h$, exactly as we found for planar motion under a central force (Week 3). From (3) we have (after integrating twice)

$$
z(t) = z(0) + \dot{z}(0)t - \frac{gt^2}{2}.
$$

Now back to (1). For a cylinder, $\rho = b =$ constant radius. Thus $\ddot{\rho} = 0$ and we have $-mb\dot{\theta}^2 = -R$. Now $h = \rho^2\dot{\theta} = b^2\dot{\theta}$, so that $\dot{\theta} = h/b^2$ and we obtain $\theta = ht/b^2 + \theta(0)$ and $R = mh^2/b^3$. Thus the reaction force has a constant magnitude and so particle stays on the wall for all time (else reaction force would become zero as it drops off). To summarise

$$
\begin{aligned}
\rho(t) &= b \\
\theta(t) &= \frac{ht}{b^2} + \theta(0) \pmod{2\pi} \\
z(t) &= z(0) + \dot{z}(0)t - \frac{gt^2}{2}
\end{aligned}
$$

Motion on surface of revolution: angular momentum conservation

We refer to figure 3. We have $m\underline{\ddot{r}} = \underline{R} - mg\underline{k}$. Hence taking the vector cross product of both sides with $\underline{r}$ we obtain $m\underline{r} \times \underline{\ddot{r}} = \underline{r} \times \underline{R} - mg\underline{r} \times \underline{k}$. Next we take the dot product of both side with $\underline{k}$ to obtain $m\underline{k} \cdot (\underline{r} \times \underline{\ddot{r}}) = \underline{k} \cdot (\underline{r} \times \underline{R}) - mg\underline{k} \cdot (\underline{r} \times \underline{k}) = \underline{k} \cdot (\underline{r} \times \underline{R})$. Now we note that $\underline{r}, \underline{k}, \underline{R}$

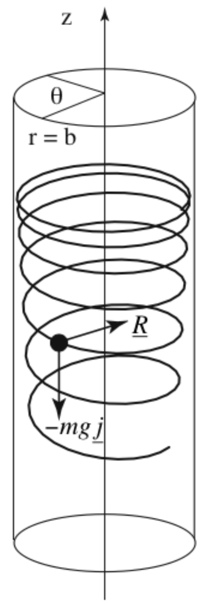

Figure 2: Path of a particle moving on the inner surface of a cylinder under gravity

are coplanar (this is due to the symmetry around the vertical axis of the surface of revolution [see figure 3]) and hence $\underline{k} \cdot (\underline{r} \times \underline{R}) = 0$. Thus $m\underline{k} \cdot (\underline{r} \times \ddot{\underline{r}}) = 0$. But then

$$\frac{d}{dt}\left\{m\underline{k} \cdot (\underline{r} \times \dot{\underline{r}})\right\} = m\underline{k} \cdot (\dot{\underline{r}} \times \dot{\underline{r}}) + m\underline{k} \cdot (\underline{r} \times \ddot{\underline{r}}) = m\underline{k} \cdot (\underline{r} \times \ddot{\underline{r}}) = 0. \tag{4}$$

We recognise $m\underline{r} \times \dot{\underline{r}}$ as angular momentum about the origin and $m\underline{k} \cdot (\underline{r} \times \dot{\underline{r}})$ as its component in the direction of $\underline{k}$, i.e. along the axis of revolution. Thus equation (4) says that the component of angular momentum in the direction of the axis of revolution is conserved.

Motion on surface of revolution: energy conservation

We have $m\ddot{\underline{r}} = \underline{F}(\underline{r})$. Thus

$$m\dot{\underline{r}} \cdot \ddot{\underline{r}} = \dot{\underline{r}} \cdot \underline{F}(\underline{r}) = \underline{F}(\underline{r}) \cdot \frac{d\underline{r}}{dt}.$$

Integrating w.r.t. t:

$$\frac{1}{2}m|\dot{\underline{r}}|^2 = \int \underline{F}(\underline{r}) \cdot \frac{d\underline{r}}{dt}\, dt + E = \int \underline{F}(\underline{r}) \cdot d\underline{r} + \text{constant}.$$

Thus we obtain conservation of energy,

$$\underbrace{\frac{1}{2}m|\dot{\underline{r}}|^2}_{\text{kinetic energy}} + \underbrace{\int -\underline{F}(\underline{r}) \cdot d\underline{r}}_{\text{potential energy}} = \text{constant}.$$

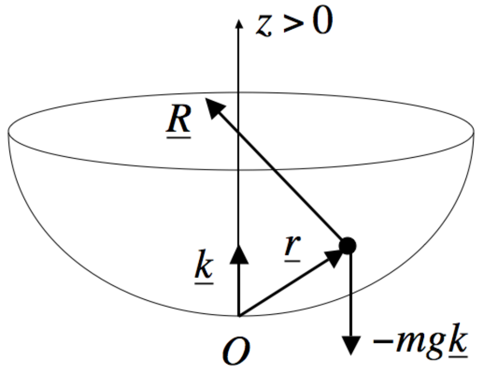

Figure 3: Angular momentum conservation. The reaction force R passes through the axis of symmetry (the z axis). Thus the vectors $\underline{k}, \underline{R}, \underline{r}$ are coplanar.

When the force $\underline{F}$ is conservative, so that $\underline{F} = -\nabla V$ for some potential function V we have $\int -\underline{F}(\underline{r}) \cdot d\underline{r} = \int \nabla V(\underline{r}) \cdot d\underline{r} = V(\underline{r}) + \text{constant}$, so the energy equation becomes

$$\underbrace{\frac{1}{2}m|\dot{\underline{r}}|^2}_{\text{kinetic energy}} \quad + \quad \underbrace{V(\underline{r})}_{\text{potential energy}} \quad = \text{constant}.$$

Example: motion on inner surface of cylinder - energy conservation

We have $\frac{d}{dt}\underline{r} = \dot{\underline{r}} = \dot{\rho}\hat{\underline{r}} + \rho\dot{\theta}\hat{\underline{\theta}} + \dot{z}\underline{k}$ so that $d\underline{r} = d\rho\hat{\underline{r}} + \rho d\theta\hat{\underline{\theta}} + dz\underline{k}$ and $\underline{F} = -R\hat{\underline{r}} - mg\underline{k}$. Thus

$$\begin{aligned}
\int \underline{F}(\underline{r}) \cdot d\underline{r} &= \int (-R\hat{\underline{r}} - mg\underline{k}) \cdot (d\rho\hat{\underline{r}} + \rho d\theta\hat{\underline{\theta}} + dz\underline{k}) \\
&= \int (-R\hat{\underline{r}} - mg\underline{k}) \cdot (d\rho\hat{\underline{r}} + \rho d\theta\hat{\underline{\theta}} + dz\underline{k}) \\
&= \int (-R\hat{\underline{r}} - mg\underline{k}) \cdot (\rho d\theta\hat{\underline{\theta}} + dz\underline{k}) \quad \text{since } d\rho = 0 \text{ on cylinder} \\
&= \int -mg\, dz = -mgz + \text{constant}
\end{aligned}$$

Now $|\dot{\underline{r}}|^2 = \dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2$, so that the energy equation becomes

$$\frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2) + mgz = \text{constant}. \tag{5}$$

What is happening here is that $\int \underline{F}(\underline{r}) \cdot d\underline{r}$ is work done by the force $\underline{F}$. The reaction force $\underline{R}$ is always normal to the surface. The velocity vector $\dot{\underline{r}}$ is tangent to the surface and hence at right angles to the reaction force. Thus $\underline{R} \cdot \dot{\underline{r}} = 0$ which gives $\int \underline{R} \cdot d\underline{r} = 0$, i.e. that the reaction force does no work. Thus we'll always get the same form (5) for the energy equation for a particle on a surface of revolution.

Example: motion on inner surface of a cone

Consider the motion of a particle on the upper surface of a cone (figure 4). Initially the particle is that $z = z_0$ and radius $\rho = b$ at $t = 0$ and is moving horizontally with speed U.

The cone equation is taken to be $\rho = kz$, so that if α is the angle shown, $\tan\alpha = k$.

The energy equation is the same as (5): $\frac{1}{2}m(\dot\rho^2 + \rho^2\dot\theta^2 + \dot z^2) + mgz = \text{constant}$. We know that the angular momentum along the z axis is conserved:

$$\text{constant} = m\underline{k}\cdot(\underline{r}\times\dot{\underline{r}}) = m\underline{k}\cdot((\rho\hat{\underline{r}} + z\underline{k})\times(\dot\rho\hat{\underline{r}} + \rho\dot\theta\hat{\underline{\theta}} + \dot z\underline{k})) = m\rho^2\dot\theta\underline{k}.$$

So lets put $h = \rho^2\dot\theta$. We may find h from the initial conditions: $t = 0$, $\rho = b$, $\rho(0)\dot\theta(0) = U$,

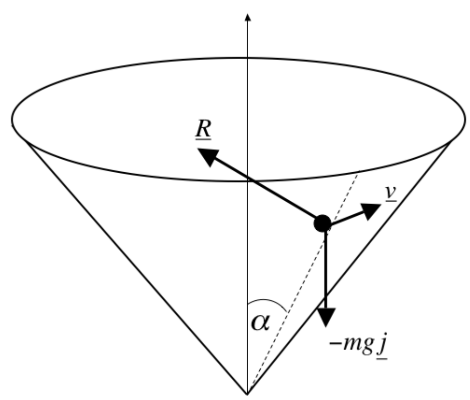

Figure 4: Motion on upper surface of a cone

i.e. $b\dot\theta(0) = U$. This gives $h = \rho^2\dot\theta = b^2\dot\theta(0) = bb\dot\theta(0) = bU$. Thus $h = \rho^2\dot\theta = bU$ and so $\dot\theta = \dfrac{h}{\rho^2} = \dfrac{bU}{\rho^2}$. From the energy equation we thus obtain (substituting for $\dot\theta$):

$$\frac{1}{2}m\left(\dot\rho^2 + \left(\frac{bU}{\rho}\right)^2 + \dot z^2\right) + mgz = \text{constant} = E, \text{ say.}$$

Now we must find the constant E. Initially $z(0) = z_0$ and $\dot\rho(0) = 0$ so $\dot z(0) = 0$ so that $E = mgz_0 + \frac{m}{2}U^2$ (since at $t = 0$ we have $|\dot{\underline{r}}|^2 = U^2$).

Now we need to use the equation for the cone $\rho = kz$ to turn the energy equation into an equation for z. We have $\dot\rho = k\dot z$ and hence

$$\frac{1}{2}m\left((1+k^2)\dot z^2 + \left(\frac{bU}{kz}\right)^2\right) + mgz = mgz_0 + \frac{1}{2}mU^2.$$

Thus the energy equation (with the m cancelled through) is

$$\frac{1}{2}\left((1+k^2)\dot z^2 + \left(\frac{bU}{kz}\right)^2\right) + gz = gz_0 + \frac{1}{2}U^2.$$

Thus we have

$$\frac{1}{2}(1+k^2)\dot{z}^2 = gz_0 + \frac{1}{2}U^2 - \frac{1}{2}\left(\frac{bU}{kz}\right)^2 - gz.$$

Now, as the particle moves on the cone this equation is satisfied, and since $\dot{z}^2 \geq 0$, its height at time t, $z(t)$, must therefore satisfy the inequality

$$gz_0 + \frac{1}{2}U^2 - \frac{1}{2}\left(\frac{bU}{kz(t)}\right)^2 - gz(t) \geq 0.$$

Let $f(z) = gz_0 + \frac{1}{2}U^2 - \frac{1}{2}\left(\frac{bU}{kz}\right)^2 - gz$ and plot the function f for $z \geq 0$ (see figure 5). As

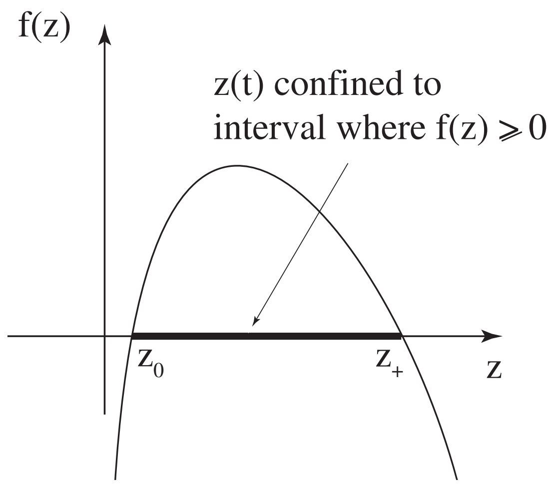

Figure 5: Constrained motion of particle in cone. For all t we must have $z(t) \in [z_0, z_+]$. Note that there is another branch to the curve for $z < 0$ not shown.

$z \to 0-$ we see that $f(z) \to -\infty$ since $\frac{bU}{kz} \to \infty$. On the other hand, as $z \to \infty$, the $-gz$ terms dominates and the $\frac{bU}{kz}$ goes to zero, the result being $f(z) \to -\infty$ as $z \to \infty$. To find the zeros of f we solve $f(z) = 0$, i.e.

$$-g(z - z_0) + \frac{U^2}{2} - \frac{U^2 b^2}{2k^2 z^2} = 0$$

But $kz_0 = b$ so that

$$-g(z - z_0) + \frac{U^2}{2} - \frac{U^2 z_0^2}{2z^2} = 0$$

which tidies to

$$(z - z_0)\left(-gz^2 + \frac{U^2}{2}(z + z_0)\right) = 0$$

Hence we see that f has zeros at

$$z = z_0, z_+ = \frac{U^2}{4g}\left(1 + \sqrt{1 + \frac{8gz_0}{U^2}}\right), z_- = \frac{U^2}{4g}\left(1 - \sqrt{1 + \frac{8gz_0}{U^2}}\right).$$

Clearly $z_- < 0$, so that the only feasible roots are z_0, z_+. Notice that since we have shown that $f(z) \to -\infty$ as $z \to 0$ and $z \to \infty$, and that there are 2 positive zeros of f, the curve must go above the horizontal $z-$axis between the two zeros. Thus we have the picture shown in figure 5. Since $0 \leq \dot{z}^2 = f(z)$, the motion of the particle msut be confined to the interval between z_0 and z_+. The only thing we haven't checked is that $z_+ > z_0$, and in fact this isn't necessarily the case: z_0 and z_+ could be swapped around. In fact this depends on the initial velocity U and position of the bead. We have $z_+ > z_0$ if and only if

$$\left(z_0 - \frac{U^2}{4g}\right)^2 - \frac{U^4}{16g^2}\left(1 + \frac{8gz_0}{U^2}\right) < 0.$$

Expanding and simplifying gives

$$z_0\left(1 - \frac{z_0U^2}{g}\right) < 0.$$

Hence $z_+ > z_0$ if and only if $U^2 > \dfrac{kg}{b}$.

General shape $z = k(\rho)$

We have the same energy equation

$$\frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2) + mgz = E,$$

and the conservation of the component of momentum along the z axis: $\rho^2\dot{\theta} = h$. If the shape of the surface is $z = k(\rho)$ then $\dot{z} = k'(\rho)\dot{\rho}$. Thus $(\rho\dot{\theta})^2 = h^2/\rho^2$ and hence

$$\frac{1}{2}m\left(\dot{\rho}^2 + \frac{h^2}{\rho^2} + k'(\rho)^2\dot{\rho}^2\right) + mgk(\rho) = E,$$

and hence

$$(1 + k'(\rho)^2)\dot{\rho}^2 + 2gk(\rho) + \frac{h^2}{\rho^2} = \frac{2E}{m}.$$

This means that during the motion the radius $r(t)$ must satisfy

$$\frac{2E}{m} - 2gk(\rho(t)) - \frac{h^2}{\rho(t)^2} \geq 0.$$

Depending on the particular shape $k(\rho)$, this will put some constraints on the motion of the particle. Working with ρ rather than z can be useful when k cannot be inverted to get a single-valued relation $\rho = k^{-1}(z)$ (a step which is needed to eliminate ρ).

Example: Motion in a cone with additional mass

Consider the set-up in figure 6. A particle P mass m on the upper surface of a frictionless cone is joined by a light string length l that passes through a hole O at the apex of the cone and supports at its other end a second particle Q mass M that hangs below the cone under gravity.

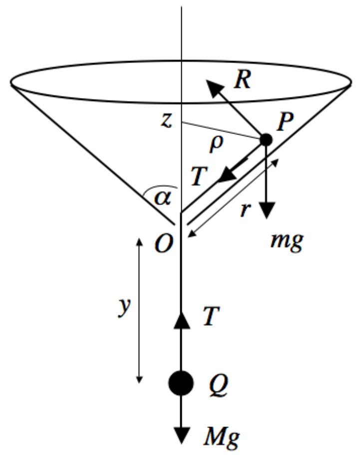

Figure 6: Particle mass m on upper surface of a cone linked by a light string to a second mass M.

Initially the particle P is projected with speed V perpendicular to OP, where $|OP| = a < l$. Assume that thereafter the string remains taut.

Let the tension in the string be T and the reaction force be R. Take z the height of particle m above O and y the distance of particle M from O. The equations of motion read

$$m\left((\ddot{\rho} - \rho\dot{\theta}^2)\underline{\hat{r}} + \frac{1}{\rho}\frac{d}{dt}(\rho^2\dot{\theta})\underline{\hat{\theta}} + \ddot{z}\underline{k}\right) = (-T\sin\alpha - R\cos\alpha)\underline{\hat{r}} + (R\sin\alpha - T\cos\alpha)\underline{k}$$

Hence for the particle of mass m,

$$m(\ddot{\rho} - \rho\dot{\theta}^2) = -T\sin\alpha - R\cos\alpha \qquad (6)$$

$$m\frac{1}{\rho}\frac{d}{dt}(\rho^2\dot{\theta}) = 0 \qquad (7)$$

$$m\ddot{z} = R\sin\alpha - T\cos\alpha - mg. \qquad (8)$$

For the particle of mass M,

$$M\ddot{y} = Mg - T. \qquad (9)$$

We also have

$$l = r + y = y + \frac{\rho}{\sin\alpha}.$$

Thus $\sin\alpha\ddot{y} = -\ddot{\rho}$ and $\ddot{z} = -\ddot{y}\cos\alpha$, since $\ddot{\rho} = \tan\alpha\ddot{z}$. From (6) and (8),

$$m\sin\alpha(\ddot{\rho} - \rho\dot{\theta}^2) + m\cos\alpha\ddot{z} = -T - mg\cos\alpha,$$

From (9)

$$m\sin\alpha(\ddot{\rho} - \rho\dot{\theta}^2) + m\cos\alpha\ddot{z} = M(\ddot{y} - g) - mg\cos\alpha.$$

Hence, using that $h = \rho^2\dot{\theta}$,

$$m\sin\alpha\left(\ddot{\rho} - \frac{h^2}{\rho^3}\right) + m\cos\alpha\frac{\ddot{\rho}}{\tan\alpha} = -M\left(\frac{\ddot{\rho}}{\sin\alpha} + g\right) - mg\cos\alpha.$$

Tidying up gives

$$(m+M)\ddot{\rho} - \frac{mh^2 \sin^2 \alpha}{\rho^3} = -Mg\sin\alpha - mg\sin\alpha\cos\alpha.$$

Circular motion $\rho = a$ is possible if $\ddot{\rho} = 0$:

$$-\frac{mh^2 \sin^2 \alpha}{a^3} = -Mg\sin\alpha - mg\sin\alpha\cos\alpha \tag{10}$$

Now $h = aV$ and so for circular motion

$$V^2 = \frac{ga(M + m\cos\alpha)}{m\sin\alpha}$$

Now suppose that the particle m moves in this circle and consider a small perturbation $\rho = a+x$, where x is small in comparison to a, that preserves h. We have

$$(m+M)\frac{d^2}{dt^2}(a+x) - \frac{mh^2 \sin^2 \alpha}{(a+x)^3} = -Mg\sin\alpha - mg\sin\alpha\cos\alpha,$$

Thus

$$(m+M)\ddot{x} - \frac{mh^2 \sin^2 \alpha}{a^3(1+(x/a))^3} = -Mg\sin\alpha - mg\sin\alpha\cos\alpha,$$

Now

$$\left(1+\frac{x}{a}\right)^{-3} = 1 - 3\frac{x}{a} + O(x^2),$$

Hence

$$(m+M)\ddot{x} - \frac{mh^2 \sin^2 \alpha}{a^3}\left(1 - 3\frac{x}{a}\right) = -Mg\sin\alpha - mg\sin\alpha\cos\alpha + O(x^2),$$

so that

$$(m+M)\ddot{x} + \frac{3mh^2 \sin^2 \alpha}{a^4}x = \frac{mh^2 \sin^2 \alpha}{a^3} - Mg\sin\alpha - mg\sin\alpha\cos\alpha + O(x^2),$$

But from (10) we obtain

$$\ddot{x} + \frac{3mh^2 \sin^2 \alpha}{(m+M)a^4}x = 0$$

to first order. Since $\frac{3mh^2 \sin^2 \alpha}{(m+M)a^4} > 0$ the circular motion is stable to small perturbations.