Math 1302, Week 3
Polar coordinates and orbital motion

1 Motion under a central force

We start by considering the motion of the earth E around the (fixed) sun (figure 1). The key point here is that the force (here gravitation) is directed towards the fixed sun. Taking the origin O at the sun and $\underline{r}$ as the position vector of the earth, the gravitation pull acts in the direction EO, i.e. in the same direction as $-\underline{r}$. One could try to write the equations of motion

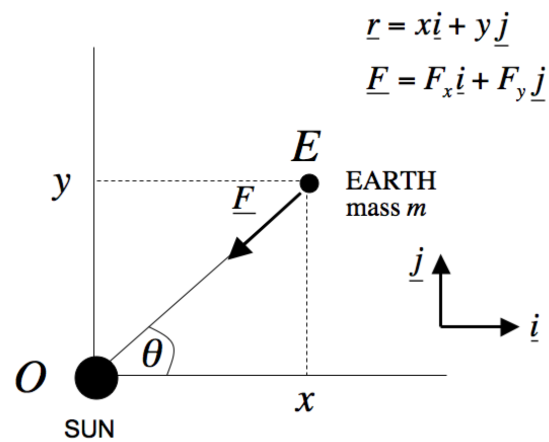

Figure 1: Motion round sun under influence of gravity

in cartesian form:

$$m\ddot{\underline{r}} = \underline{F}$$

becomes

$$m(\ddot{x}\underline{i} + \ddot{y}\underline{j}) = F_x\underline{i} + F_y\underline{j}.$$

Since the force is central (towards O),

$$\underline{F} = F\frac{1}{|\underline{r}|}(-\underline{r}),$$

where $F = |\underline{F}|$ and $\underline{r} = \underline{OE}$. Now

$$\begin{aligned} F_x &= \underline{F}\cdot\underline{i} = -F\cos\theta = -F\times\frac{x}{\sqrt{x^2+y^2}} \\ F_y &= \underline{F}\cdot\underline{j} = -F\sin\theta = -F\times\frac{y}{\sqrt{x^2+y^2}} \end{aligned}$$

Hence the equations of motion read

$$\ddot{x} = -\frac{xF(\underline{r})}{m\sqrt{x^2+y^2}} \quad (1)$$

$$\ddot{y} = -\frac{yF(\underline{r})}{m\sqrt{x^2+y^2}} \quad (2)$$

where now I have replaced F by $F(\underline{r})$ to emphasise that F varies with x, y. It is not immediately obvious how to tackle these equations! The cartesian description does not lend itself ideally to this model.

To deal with motion under a central force it pays dividends to use a different coordinate system. The force points along the position vector, so a natural way to write any vector is in terms of a unit vector $\underline{\hat{r}}$ pointing along $\underline{r}$ and another unit vector $\underline{\hat{\theta}}$ pointing perpendicular to $\underline{\hat{r}}$ (see figure 2). The force $\underline{F}$ then takes the very simple form[1]

$$\underline{F} = |\underline{F}|(-\underline{\hat{r}}).$$

To apply Newton's law, the problem now becomes how to express the acceleration $\underline{a}$ in terms

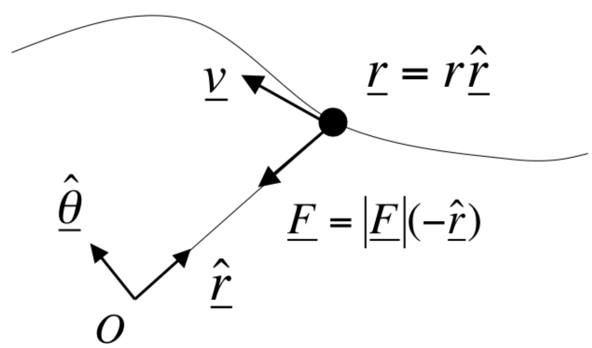

Figure 2: Polar coordinates: choice of orthonormal basis vectors

of the basis vectors $\underline{\hat{r}}, \underline{\hat{\theta}}$, *and these basis vectors are different at different points in space*! Thus although we have made the force easier to deal with, the acceleration is trickier to calculate...but the effort, it turns out, is worth it, since Newton's law will immediately tell us that the component of acceleration in the direction of $\underline{\hat{\theta}}$ is *zero*, since the force acts purely along $\underline{\hat{r}}$. This will tell us that the angular momentum is conserved.

2 Acceleration in polar coordinates

Consider figure 3 where a particle P is at $\underline{r} = x\underline{i} + y\underline{j}$ in cartesian coordinates. Let $r = |\underline{r}|$ be the distance of the particle from the origin and $\underline{\hat{r}} = \underline{r}/r$ the unit vector pointing along OP. Let

[1] $|\underline{F}|(-\underline{\hat{r}})$ for an attractive force, $|\underline{F}|\underline{\hat{r}}$ for a repulsive force

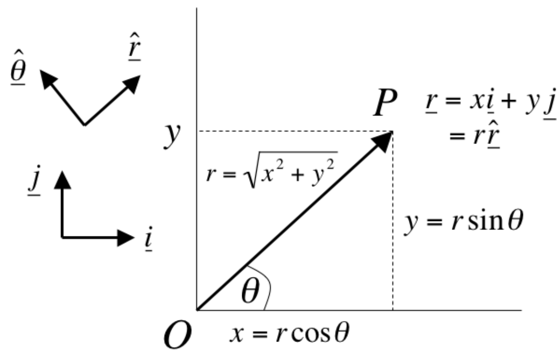

Figure 3: Polar coordinates

$\hat{\underline{\theta}}$ be a unit vector perpendicular to $\hat{\underline{r}}$ as shown. Then

$$\underline{r} = r\hat{\underline{r}},$$

so that the velocity of P is

$$\dot{\underline{r}} = \dot{r}\hat{\underline{r}} + r\dot{\hat{\underline{r}}}.$$

So we need $\dot{\hat{\underline{r}}}$. But $\hat{\underline{r}} = \cos\theta\underline{i} + \sin\theta\underline{j}$, so that $\dot{\hat{\underline{r}}} = -\sin\theta\dot{\theta}\underline{i} + \cos\theta\dot{\theta}\underline{j} = (-\sin\theta\underline{i} + \cos\theta\underline{j})\dot{\theta}$. Now by the geometry, $-\sin\theta\underline{i} + \cos\theta\underline{j} = \hat{\underline{\theta}}$. Thus $\dot{\hat{\underline{r}}} = \hat{\underline{\theta}}\dot{\theta}$ and we get

$$\boxed{\dot{\underline{r}} = \dot{r}\hat{\underline{r}} + r\dot{\theta}\hat{\underline{\theta}}}$$

Thus the transverse (tangential) velocity component is $r\dot{\theta}$ and the radial component is $\dot{r}$.
Now for the acceleration. Differentiate again:

$$\ddot{\underline{r}} = \ddot{r}\hat{\underline{r}} + \dot{r}\frac{d}{dt}\hat{\underline{r}} + \frac{d}{dt}(r\dot{\theta})\hat{\underline{\theta}} + r\dot{\theta}\frac{d}{dt}\hat{\underline{\theta}} = \ddot{r}\hat{\underline{r}} + \dot{r}\hat{\underline{\theta}}\dot{\theta} + (\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\underline{\theta}} + r\dot{\theta}\frac{d}{dt}\hat{\underline{\theta}}.$$

To complete the formula, we thus need

$$\frac{d}{dt}\hat{\underline{\theta}} = \frac{d}{dt}(-\sin\theta\underline{i} + \cos\theta\underline{j}) = -\cos\theta\dot{\theta}\underline{i} - \sin\theta\dot{\theta}\underline{j} = -\hat{\underline{r}}\dot{\theta}$$

and hence we have $\ddot{\underline{r}} = \ddot{r}\hat{\underline{r}} + \dot{r}\hat{\underline{\theta}}\dot{\theta} + (\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\underline{\theta}} - r\dot{\theta}^2\hat{\underline{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{\underline{r}} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\underline{\theta}}$. Sometimes it is convenient to rewrite $2\dot{r}\dot{\theta} + r\ddot{\theta} = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$. To summarise:

$$\boxed{\begin{aligned} \text{Velocity: } \dot{\underline{r}} &= \dot{r}\hat{\underline{r}} + r\dot{\theta}\hat{\underline{\theta}} \\ \text{Acceleration: } \ddot{\underline{r}} &= \underbrace{(\ddot{r} - r\dot{\theta}^2)}_{\text{radial component}}\hat{\underline{r}} + \underbrace{\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})}_{\text{tangential component}}\hat{\underline{\theta}}. \end{aligned}}$$

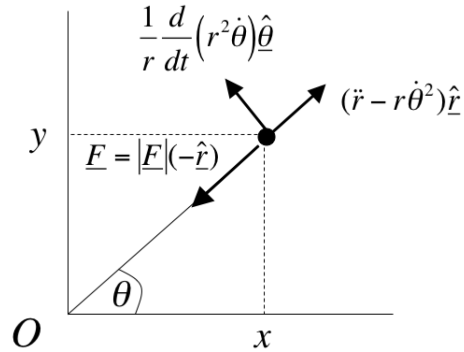

Figure 4: Motion under an attractive central force in the plane

3 Motion under a central force in polar coordinates

Now instead of using $-|\underline{F}|\hat{\underline{r}}$ for an attractive central force and $+|\underline{F}|\hat{\underline{r}}$ for a repulsive central force we will write the force as $f(\underline{r})\hat{\underline{r}}$, where $f < 0$ for attractive and $f > 0$ for repulsive forces.

If a particle mass m moves in the plane under an attractive force $\underline{F}$ we have $m\ddot{\underline{r}} = \underline{F}$ (see figure 4). Suppose that $\underline{F} = f(\underline{r})\hat{\underline{r}}$. We have for such a **central force**,

$$m\ddot{\underline{r}} = m(\ddot{r} - r\dot{\theta}^2)\hat{\underline{r}} + \frac{m}{r}\frac{d}{dt}(r^2\dot{\theta})\hat{\underline{\theta}} = f(\underline{r})\hat{\underline{r}}.$$

Since $\hat{\underline{r}}, \hat{\underline{\theta}}$ are an orthonormal pair of vectors, we may equate coefficients of $\hat{\underline{r}}, \hat{\underline{\theta}}$ on both side to obtain

$$\boxed{\begin{array}{rcl} m(\ddot{r} - r\dot{\theta}^2) & = & f(\underline{r}) \\ \frac{m}{r}\frac{d}{dt}(r^2\dot{\theta}) & = & 0. \end{array}} \qquad (1a, b)$$

The second equation (1b) tells us the very important fact: $r^2\dot{\theta}$ is conserved during the motion. It is traditional to write $r^2\dot{\theta} = h$ where h is constant (it is angular momentum per unit mass). Thus (1b) leads to conservation of angular momentum, i.e. $r^2\dot{\theta} = h$, and this results from the fact that the force is central (if not central, there would be a non-zero component of $\hat{\underline{\theta}}$ in the force, so that the right hand side of (1b) would be non-zero and $r^2\dot{\theta}$ would no longer be constant).

From (1a,b) have $m(\ddot{r} - r\dot{\theta}^2) = f(\underline{r})$ and $r^2\dot{\theta} = h$. Rearranging the second equation, $\dot{\theta} = h/r^2$, so that

$$m(\ddot{r} - \frac{h^2}{r^3}) = f(\underline{r}).$$

Unlike when we used cartesian coordinates, where the x, y coordinates were tied up in two equations (1) and (2), we now have just one equation in r.

1 Conservation of Energy

From Newton's second law,

$$m\ddot{\underline{r}} = \underline{F}(\underline{r}).$$

Thus

$$\int m\dot{\underline{r}} \cdot \ddot{\underline{r}}\, dt = \int \dot{\underline{r}} \cdot \underline{F}(\underline{r})\, dt = \int \underline{F}(\underline{r}) \cdot \frac{d\underline{r}}{dt}\, dt = \int \underline{F}(\underline{r}) \cdot d\underline{r}.$$

so that

$$\frac{1}{2}m|\dot{\underline{r}}|^2 - \int \underline{F}(\underline{r}) \cdot d\underline{r} = \text{constant}.$$

If the force $\underline{F}$ is conservative (so that the work done in going between two points is independent of the path), then there is a potential function V such that $\underline{F} = -\nabla V$ and we have from the previous equation

$$\underbrace{\frac{1}{2}m|\dot{\underline{r}}|^2}_{\text{kinetic energy}} + \underbrace{V(\underline{r})}_{\text{potential energy}} = E,$$

where E is the total conserved energy. Since $\underline{r} = \dot{r}\hat{\underline{r}} + r\dot{\theta}\hat{\underline{\theta}} = \dot{r}\hat{\underline{r}} + \frac{h}{r}\hat{\underline{\theta}}$ we have

$$\boxed{\frac{m\dot{r}^2}{2} + \frac{mh^2}{2r^2} + V(r) = E.}$$

Here E is the conserved total energy. When the force is gravitational, with gravitational constant G we get $V(\underline{r}) = -\dfrac{GMm}{r}$.

A very useful trick is to change from using r to using $u = 1/r$. With this change we find:

$$\frac{du}{d\theta} = \frac{dt}{d\theta}\frac{du}{dt} = \frac{1}{\dot{\theta}}\frac{d}{dt}\left(\frac{1}{r}\right) = \frac{1}{\dot{\theta}}\left(\frac{-1}{r^2}\right)\dot{r} = -\frac{\dot{r}}{h}. \tag{4}$$

Differentiation w.r.t. θ yields:

$$\frac{d^2u}{d\theta^2} = \frac{dt}{d\theta}\frac{d}{dt}\left(\frac{-\dot{r}}{h}\right) = -\frac{1}{h\dot{\theta}}\ddot{r}.$$

Thus $\ddot{r} = -h\dot{\theta}\frac{d^2u}{d\theta^2} = -h(hu^2)\frac{d^2u}{d\theta^2} = -h^2u^2\frac{d^2u}{d\theta^2}$, since $h = r^2\dot{\theta} = \dot{\theta}/u^2$ gives $\dot{\theta} = hu^2$. Hence from (1a),

$$m\left(-h^2u^2\frac{d^2u}{d\theta^2} - (1/u)(hu^2)^2\right) = f(1/u)$$

which rearranges to give

$$\boxed{\frac{d^2u}{d\theta^2} + u = -\frac{f(1/u)}{mh^2u^2}}. \tag{5}$$

4 Planetary motion

We wish to establish the following laws that Johannes Kepler (1605) extracted from the observations of Tycho Brahe:

> **Kepler's Laws of motion**
>
> **First Law** Planets move in ellipses with the sun at a focus;;
>
> **Second Law** The area swept out per unit time by the radius vector joining a planet and the sun is constant;
>
> **Third Law** The ratio of the square of the period of orbit to the cube of the semi-major axis is constant.

The planets orbit the sun constrained planes that pass through the sun (see Question sheet 3, Qu 2). Thus we may describe the position of a planet using polar coordinates in the appropriate plane, and the motion is given by (1a,b). Often it is more convenient to work with equation (5). When the force is due to gravity, we have $f(r) = -GmM/r^2 = -\mu/r^2$ where $\mu = GmM$ and $G > 0$ is the gravitational constant, m is mass of the planet and M the mass of the Sun. Equation (5) then simplifies to

$$\frac{d^2u}{d\theta^2} + u = -\frac{(-\mu u^2)}{mh^2u^2} = \frac{\mu}{mh^2}.$$

The general solution of this is $u = C\cos\theta + D\sin\theta +$ particular integral, and it is easy to see that the particular integral is $\frac{\mu}{mh^2}$. Thus

$$u = C\cos\theta + D\sin\theta + \frac{\mu}{mh^2}.$$

Alternatively we can write the general solution more conveniently as

$$u = \frac{1}{r} = A\cos(\theta - \delta) + \frac{\mu}{mh^2}.$$

Now define

$$\boxed{\ell = \frac{mh^2}{\mu}, e = \frac{Amh^2}{\mu},}$$

so that the previous equation becomes

$$\boxed{\frac{\ell}{r} = 1 + e\cos(\theta - \delta).} \tag{6}$$

Equation (6) is the equation[2] for a conic section in polar coordinates, and you need to know that

$$\boxed{\begin{array}{rl} 0 < e < 1 & \text{ellipse} \\ e = 0 & \text{circle} \\ e = 1 & \text{parabola} \\ e > 1 & \text{hyperbola} \end{array}} \tag{7}$$

[2]Normally in geometry textbooks the formula is given when $\delta = 0$. The effect of δ is to rotate the curve by δ about the origin.

It is important to appreciate that the radius r in (6) is the distance of a point on the curve from a focus. You are referred to the Appendix for a discussion of the conic sections for various eccentricities $e > 0$.

For the planets $f(r) = -\mu/r^2$ where $\mu = GmM$, G is the gravitational constant, m is the mass of the planet, M the mass of the Sun. We find that

$$\ell = \frac{mh^2}{\mu} = \frac{h^2}{GM}, \quad e = \frac{Amh^2}{\mu} = \frac{Amh^2}{GMm} = \frac{Ah^2}{GM}.$$

One finds that for all the planets in the solar system that $e < 1$ (in fact most of them have $e \ll 1$ so that their motion is close to circular). Thus from (7) we have Kepler's first law that the planets move in ellipse with the Sun at a focus.

For the second law, suppose that in time T the radius vector sweeps out an area A as θ changes from θ_0 to θ_T. We use that $r^2\dot{\theta} = h$ so that (see figure 5)

$$A = \int_{\theta_0}^{\theta_T} \frac{1}{2} r(\theta)^2\, d\theta = \int_0^T \frac{1}{2} r(t)^2 \frac{d\theta(t)}{dt}\, dt = \frac{1}{2} hT.$$

Thus the area swept out by the radius per unit time is $A/T = h/2$ a constant. This is Kepler's second law.

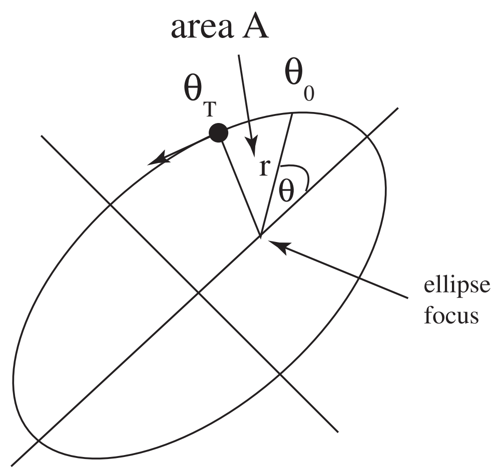

Figure 5: Kepler's second law

Finally for the third law. We have, from Kepler's second law,

$$\frac{\text{Area of ellipse}}{\text{Period of orbit}} = \frac{h}{2}.$$

But area of an ellipse with semi-major axis b and semi-minor axis a is πab. Hence

$$T_{period} = \frac{2\pi ab}{h}.$$

We also have $\ell = a(1 - e^2), b^2 = a^2(1 - e^2)$ and $\ell = h^2/GM$. Thus

$$T_{period}^2 = \frac{4\pi^2 a^2 b^2}{h^2} = \frac{4\pi a^2 b^2}{GM\ell} = \frac{4\pi a^2 b^2}{GMa(1 - e^2)} = \frac{4\pi a^2 b^2}{GMa(b^2/a^2)} = \frac{4\pi a^3}{GM}.$$

Thus

$$\frac{T_{period}^2}{a^3} = \frac{4\pi}{GM}$$

is independent of the planet, and hence we have Kepler's third law.

Before reading the example problems you might want to read the appendix on ellipses.

5 Some example problems for central forces

Example 1

Consider a particle moving under a central force $f(r)\underline{\hat{r}}$ per unit mass where $f(r) = k/r^3$ with $k > 0$ constant [thus force is repulsive]. Find $u(\theta)$ given that the particle is projected from the point $r = a, \theta = 0$ with radial and transverse velocities U, V respectively. Show that the particle moves off the infinity when $\tan(q\theta) \to qV/U$ where $q^2 = 1 + k/(a^2V^2)$. Solution: We have from

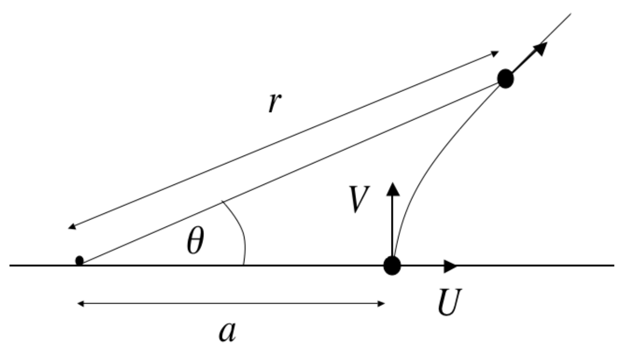

Figure 6: Example 1

equation (5) that

$$\frac{d^2u}{d\theta^2} + u = -\frac{(ku^3)}{h^2u^2} = -\frac{ku}{h^2} \Rightarrow \frac{d^2u}{d\theta^2} + \left(1 + \frac{k}{h^2}\right) u = 0.$$

This has general solution $u = A\cos\omega\theta + B\sin\omega\theta$ where $\omega^2 = 1 + k/h^2$. When faced with such a solution we need <u>two conditions</u> to get the <u>two constants</u> A, B. The first condition comes from knowing the position of the particle at some point, and the second from knowing something about its speed at some point.

Thus $\theta = 0, u^{-1} = r = a$ gives $A = 1/a$. Also $\theta = 0, \dot{r} = U, a\dot{\theta} = V$. Now we have

$$\left.\frac{du}{d\theta}\right|_{\theta=0} = -\omega A \sin\omega 0 + B\omega\cos\omega 0 = B\omega.$$

To find h we use $r^2\dot{\theta} = h$ at the point $r = a$, since then $h = a^2\dot{\theta} = a(a\dot{\theta}) = aV$. But $\dot{r} = -h\frac{du}{d\theta}$, so that at $t = 0$, $U = \dot{r} = -(aV)\frac{du}{d\theta}$. This gives that $B = -U/(\omega aV)$ and hence that

$$u(\theta) = \frac{1}{a}\cos\left(\sqrt{1+\frac{k}{a^2V^2}}\right)\theta - \frac{U}{\sqrt{k+a^2V^2}}\sin\left(\sqrt{1+\frac{k}{a^2V^2}}\right)\theta.$$

For the second part, the particle goes off to infinity when $r(\theta) \to \infty$, i.e. $u(\theta) \to 0$. But then this means $\frac{1}{a}\cos\left(\sqrt{1+\frac{k}{a^2V^2}}\right)\theta - \frac{U}{\sqrt{k+a^2V^2}}\sin\left(\sqrt{1+\frac{k}{a^2V^2}}\right)\theta \to 0$, i.e. $\tan q\theta \to \frac{\sqrt{k+a^2V^2}}{aU} = qV/U$ as required.

Example 2

Suppose particle moves under an attractive force per unit mass $\frac{\alpha}{r^2} + \frac{\beta}{r^3}$ where $\beta = \frac{a\alpha}{2}$ (i.e. a force directed towards the origin). If at $t = 0$ the particle is distance a from the origin and moving purely tangentially with speed $V = \sqrt{\alpha/a}$ find the equation for its path and the furthest and closest it comes to the origin.

Solution: We have $f(r) = -\frac{\alpha}{r^2} - \frac{\beta}{r^3}$ and so $\frac{d^2u}{d\theta^2} + u = \frac{(\alpha u^2+\beta u^3)}{h^2u^2}$ so that $\frac{d^2u}{d\theta^2} + \left(1 - \frac{\beta}{h^2}\right)u = \frac{\alpha}{h^2}$.

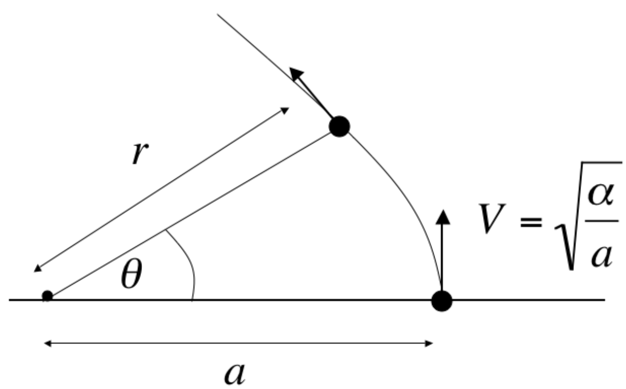

Figure 7: Example 2

Now find h. Take $\theta = 0$ where $t = 0$. Then $h = r^2\dot{\theta}\Big|_{t=0} = a^2\dot{\theta}(0) = aa\dot{\theta}(0) = aV$. Putting this value of h in the equation for u gives

$$\frac{d^2u}{d\theta^2} + \left(1 - \frac{\frac{1}{2}\alpha a}{a^2V^2}\right)u = \frac{\alpha}{a^2V^2}$$

which tidies to $\frac{d^2u}{d\theta^2} + \frac{1}{2}u = \frac{1}{a}$ which has the general solution

$$u = \frac{2}{a} + A\cos\left(\frac{\theta}{\sqrt{2}}\right) + B\sin\left(\frac{\theta}{\sqrt{2}}\right).$$

But now use $\theta = 0, r = a$ to get $\frac{1}{a} = \frac{2}{a} + A\cos 0$ so that $A = -1/a$. Also $\dfrac{du}{d\theta} = -\dot{r}h = 0$ when $t = 0$ (since the motion is then purely tangential). Thus $\frac{1}{\sqrt{2}}(B\cos(\frac{\theta}{\sqrt{2}}) - A\sin(\frac{\theta}{\sqrt{2}}))\Big|_{\theta=0} = 0$ which gives $B = 0$ and hence

$$u = \frac{1}{r} = \frac{2}{a} - \frac{1}{a}\cos\left(\frac{\theta}{\sqrt{2}}\right).$$

Thus $r(\theta) = (\frac{2}{a} - \frac{1}{a}\cos\left(\frac{\theta}{\sqrt{2}}\right))^{-1}$ and r is max when $\frac{2}{a} - \frac{1}{a}\cos\left(\frac{\theta}{\sqrt{2}}\right)$ is min, i.e. when $\theta = 0$ which corresponds to $r = a$ and r is min when $\frac{2}{a} - \frac{1}{a}\cos\left(\frac{\theta}{\sqrt{2}}\right)$ is max, i.e. when $\theta = \sqrt{2}\pi$ which is when $r = a/3$. The orbit is shown in figure 8 for $a = 1$. They are not ellipses!

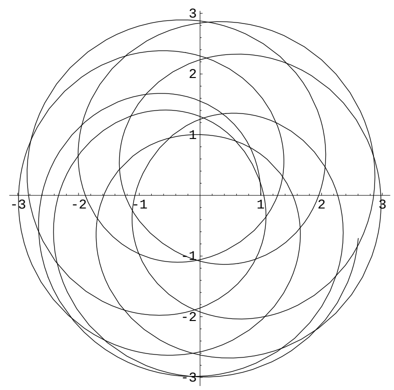

Figure 8: Orbits for example 2

6 Stability of circular orbits

First we need to find the conditions that a particle moves in a circular orbit. We take $m = 1$ for the mass. Recall that we have $\ddot{r} - \frac{h^2}{r^3} = f(r)$. For circular motion radius b we have $r = b$, $\dot{\theta} = \omega_0$ and $h = h_0$ say. Thus we have $\ddot{r} - (h^2/r^3) = 0 - h_0^2/b^3 = f(b)$ and also $ub = h_0$ where u is the constant speed $u = b\omega_0$. Hence we have for circular motion

$$\boxed{\begin{aligned} -\frac{h_0^2}{b^3} &= f(b) \\ u &= \frac{h_0}{b} \end{aligned}}$$

Now consider small perturbations of the circular orbit. Thus suppose $r = b + x$ and $h = h_0 + H$ with $x, H \ll 1$. Note that the new h of the perturbed motion is also constant by conservation of angular momentum. Thus H is also constant. From $\ddot{r} - h^2/r^3 = f(r)$ we obtain

$$\frac{d^2}{dt^2}(b + x) - \frac{(h_0 + H)^2}{(b + x)^3} = f(b + x).$$

Now

$$\frac{1}{(b + x)^3} = \frac{1}{b^3}\frac{1}{(1 + \frac{x}{b})^3} = \frac{1}{b^3}(1 - \frac{3x}{b} + \cdots),$$

and by Taylor's theorem

$$f(b+x) = f(b) + f'(b)x + \cdots .$$

Hence we have

$$\ddot{x} - \frac{1}{b^3}(h_0^2 + 2h_0 H + H^2)(1 - \frac{3x}{b} + \cdots) = f(b) + f'(b)x + \cdots .$$

Hence

$$\ddot{x} - \frac{1}{b^3}(h_0^2 + 2h_0 H - \frac{3xh_0^2}{b}) = f(b) + xf'(b) + \text{higher orders in } x, H$$

But for circular motion we had $-\frac{h_0^2}{b^3} = f(b)$ and hence $\ddot{x} - \frac{2h_0 H}{b^3} + \frac{3xh_0^2}{b^4} = xf'(b)$ to first order. Letting $q^2 = f'(b) - \frac{3h_0^2}{b^4}$ and $x_0 = \frac{2h_0 H}{b^3 q^2}$ we have $\ddot{x} - q^2(x - x_0) = 0$ which has general solution

$$x = x_0 + Ae^{qt} + Be^{-qt}.$$

If $q^2 > 0$, so that q is real, either e^{qt} or e^{-qt} grows rapidly and hence the orbit is unstable. If $q^2 < 0$ then q is complex and we get a stable "wobble" around the circular orbit:

$$\boxed{\begin{array}{ll} q^2 = f'(b) - \frac{3h_0^2}{b^4} > 0 & \Rightarrow \text{unstable} \\ q^2 = f'(b) - \frac{3h_0^2}{b^4} < 0 & \Rightarrow \text{stable} \end{array}}$$

7 Appendix: The polar equation $\ell/r = 1 + e\cos\theta$

Here we consider the geometry of

$$\frac{\ell}{r} = 1 + e\cos\theta, \tag{8}$$

for various $e \geq 0$.

Since $x = r\cos\theta$ we obtain

$$\begin{aligned} \ell &= er\cos\theta + r \\ &= ex + \sqrt{x^2 + y^2}. \end{aligned}$$

Thus

$$\begin{aligned} (\ell - ex)^2 &= x^2 + y^2 \\ \ell^2 - 2\ell ex + e^2x^2 &= x^2 + y^2 \end{aligned}$$

Rearranging we get

$$(1 - e^2)x^2 + 2\ell ex + y^2 = \ell^2,$$

and

$$\left(x + \frac{el}{1 - e^2}\right)^2 + \frac{y^2}{1 - e^2} = \frac{\ell^2}{(1 - e^2)^2},$$

Now consider various values of e.

1. $e = 0$ Here $r = \ell$. Thus the conic is a circle;

2. $0 < e < 1$ Let $a = \dfrac{\ell}{1-e^2} > 0$ and $b = \dfrac{\ell}{\sqrt{1-e^2}} > 0$. Then $\left(\dfrac{x}{a}\right)^2 + \left(\dfrac{y}{b}\right)^2 = 1$ and we get an ellipse.

3. $e = 1$ Then $y^2 = \ell^2 - 2\ell x$ and we have a parabola.

4. $e > 1$ Then with $a = \dfrac{\ell}{e^2-1} > 0$ and $b = \dfrac{\ell}{\sqrt{e^2-1}} > 0$ we have $\left(\dfrac{x}{a}\right)^2 - \left(\dfrac{y}{b}\right)^2 = 1$ which is a hyperbola.

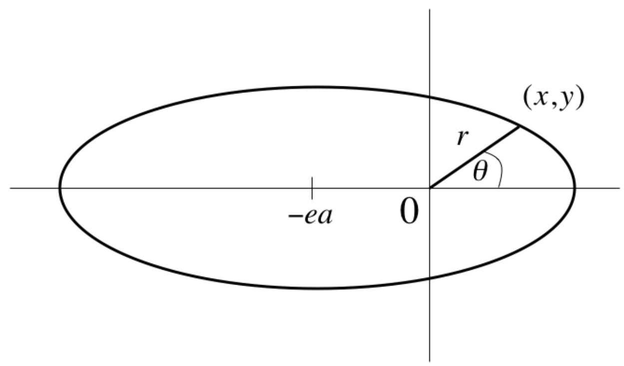

Figure 9: Geometry of an ellipse $e \in (0, 1)$.

For an introduction of conic sections, see pages 198-212 in "What is Mathematics?" by Richard Courant and Herbert Robbins, OUP (1980).