# MATH1302, Week 2

# Motion on a planar curve: intrinsic coordinates

Last week we considered the motion of a ball sliding down a fixed inclined plane. We found it was easier to treat it using basis vectors parallel and normal to the plane (i.e. resolving forces and accelerations tangent and normal to the plane). This week we start by considering the motion of a ball sliding down a curved plane (see figure 1). If we take the same approach, $\underline{e}_1, \underline{e}_2$

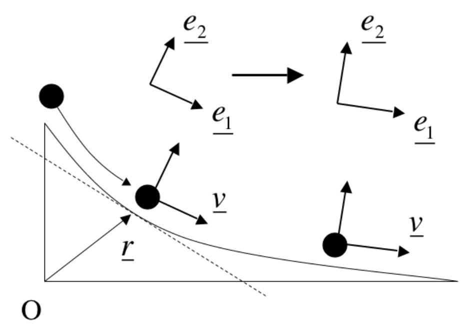


Figure 1: As ball slides down the plane the basis vectors $\underline{e}_1, \underline{e}_2$ change. The $\underline{e}_1$ is tangent to the curved plane surface. The velocity is also tangent to the curved plane and in the direction of $\underline{e}_1$

change as the ball progresses down the plane. To deal with this we'll need to introduce intrinsic coordinates, as we now do.

Thus consider a smooth curve $y = y(x)$ (see figure 2). Take as coordinates, $s =$ (signed) distance along the curve and $\psi$ the angle the tangent makes to the horizontal. Take $s = 0$ somewhere convenient (typically where $\psi = 0$) and $s$ increasing forwards along the curve and $s$ decreasing (i.e. $s < 0$) in the opposite direction. Instead of specifying the curve by $y = y(x)$ we may specify it by giving the angle $\psi$ as a function of $s$ by $\psi = \psi(s)$, $s \in \mathbb{R}$. In describing $s$ as a function of $\psi$, we may have to restrict to an interval $I$ (e.g. if $\psi = \sin^{-1}(s)$ then $s = \sin\psi$ will repeat every $2\pi$ in $s$). There are two crucial formulae (see the triangle in the figure):

$$\boxed{\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}} \tag{1}$$

and

$$\boxed{\tan\psi = \frac{dy}{dx}.} \tag{2}$$

## Example 1: $y = \cosh x$ in intrinsic coordinates

Here $dy/dx = \sinh x$, so $ds/dx = \sqrt{1 + \sinh^2 x} = \cosh x$ and hence $s(x) = \sinh x + C$, $C$ constant. Take $s = 0$ at the point $x = 0$ for convenience, so that $s(x) = \sinh x$. Thus $x(s) =$

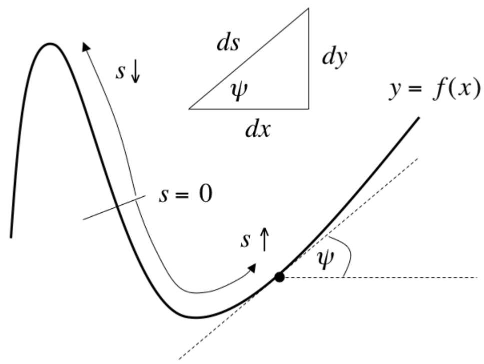


Figure 2: Use of signed arclength $s$ and angle $\psi$ as intrinsic coordinates. $\psi$ is measured from the positive $x-$axis and $s = 0$ is taken to be at some convenient point on the curve.

$\sinh^{-1} s$. We also have $y = \cosh x = \sqrt{1 + \sinh^2 x} = \sqrt{1 + s^2}$, i.e. $y(s) = \sqrt{1 + s^2}$. Finally $\tan\psi = dy/dx = \sinh x = s$, so $\psi = \tan^{-1} s$.

## Example 2: Cycloid $x = a(\theta + \sin\theta), y = a(1 - \cos(\theta))$ in intrinsic coordinates on $(-\pi, \pi)$

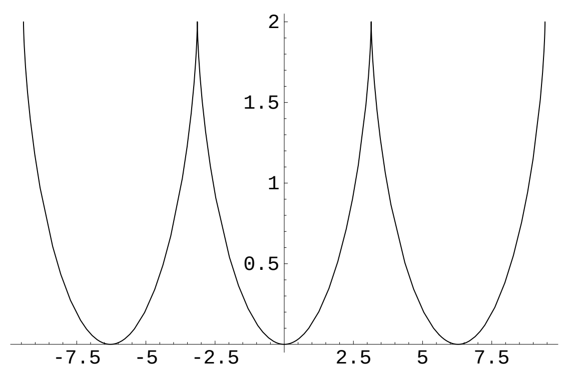


Figure 3: Cycloid given by $x = a(\theta + \sin\theta), y = a(1 - \cos\theta)$ (Here we have taken $a = 1$ for convenience).

Now the curve is given parametrically in terms of $\theta$ (see figure 3). We have

$$\tan\psi = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\sin\theta}{a(1 + \cos\theta)} = \frac{2\sin(\theta/2)\cos(\theta/2)}{2\cos^2(\theta/2)} = \tan(\theta/2).$$

Hence $\psi = \theta/2$. Moreover,

$$ds/d\theta = \sqrt{(dx/d\theta)^2 + (dy/d\theta)^2}$$

$$= a\sqrt{(1 + \cos\theta)^2 + \sin^2\theta} = a\sqrt{2 + 2\cos\theta} = a\sqrt{4\cos^2\theta/2} = 2a\cos(\theta/2).$$

Hence

$$\boxed{s = 4a\sin(\theta/2) = 4a\sin\psi.}$$

# 1 Tangents and normals to planar curves

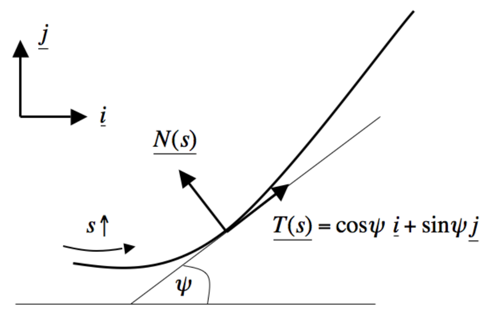


Figure 4: Unit tangent $\underline{T}$ and unit normal $\underline{N}$ to a curve.

We have $\underline{T} = \frac{dr}{ds}$ as the unit tangent to a curve parameterised by (signed) arclength. Since $\underline{T}$ is a unit normal,

$$0 = \frac{d}{ds}\left(\underline{T}\cdot\underline{T}\right) = 2\underline{T}\cdot\frac{d\underline{T}}{ds},$$

so that $\dfrac{d\underline{T}}{ds}$ is normal to the curve. But in terms of the angle $\psi$ measured from the positive $x-$axis to the tangent,

$$\underline{T} = \cos\psi\underline{i} + \sin\psi\underline{j}, \tag{3}$$

Thus

$$\frac{d\underline{T}}{ds} = (-\sin\psi\underline{i} + \cos\psi\underline{j})\frac{d\psi}{ds}.$$

We define

$$\underline{N} = -\sin\psi\underline{i} + \cos\psi\underline{j} \tag{4}$$

to be the unit normal to the curve and call

$$\boxed{\kappa(s) = \frac{d\psi}{ds} \quad \text{the curvature of the curve at } s.}$$

Note that $\kappa$ can have either sign. When $\kappa = 0$ we get a point of inflexion. With these definitions,

$$\boxed{\frac{d\underline{T}}{ds} = \kappa\underline{N}.}$$

## 2 Velocity and acceleration in instrinsic coordinates

We have

$$\frac{d\underline{T}}{ds} = \kappa\underline{N}, \quad \kappa = \frac{d\psi}{ds}$$

The pair $\underline{T}, \underline{N}$ form an orthonormal pair of vectors ($\underline{T} \cdot \underline{T} = 1, \underline{N} \cdot \underline{N} = 1, \underline{T} \cdot \underline{N} = 0$). At a distance $s$ along the curve, any vector $\underline{c}$ can be expressed as a sum $\underline{c} = \lambda\underline{T} + \mu\underline{N}$ (in fact $\lambda = \underline{c} \cdot \underline{T}, \mu = \underline{c} \cdot \underline{N}$). Hence, in particular, the velocity $\underline{v}$ and acceleration $\underline{a} = \dot{\underline{v}}$ have such a representation.

The velocity has a very simple form:

$$\boxed{\underline{v} = \dot{s}\underline{T}.} \tag{5}$$

The simplicity of equation (5) is one benefit of choosing intrinsic coordinates: the tangent vector $\underline{T}$ always points along the same line as the velocity $\underline{v}$, and the speed is just $|\dot{s}|$, the rate of covering distance $s$ along the curve (we need the modulus $|\cdot|$ since $\dot{s}$ could be negative, such as in an oscillation about a point where $s = 0$). Notice that $\underline{T}$ was fixed by (3) whereas $\underline{N}$ is fixed by (4), they are not defined in terms of mechanics on the curve.

Now for the acceleration. We need

$$\dot{\underline{v}} = \frac{d}{dt}(\dot{s}\underline{T}) = \ddot{s}\underline{T} + \dot{s}\frac{d\underline{T}}{dt}.$$

But we have

$$\frac{d\underline{T}}{dt} = \frac{d\underline{T}}{ds}\frac{ds}{dt} = \dot{s}\kappa\underline{N}.$$

Thus

$$\boxed{\dot{\underline{v}} = \ddot{s}\underline{T} + \kappa\dot{s}^2\underline{N}.}$$

We now need to find $\kappa$ in terms of $s, \psi$. We will show that $|\kappa| = |d\psi/ds| = 1/\rho$, where $\rho$ is the radius of curvature of the curve.

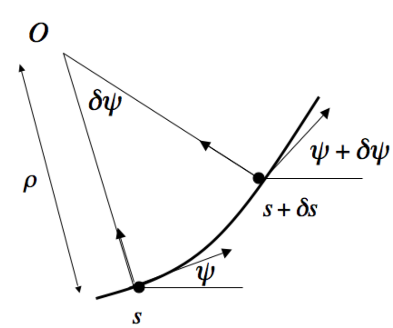


Figure 5: Radius of curvature in intrinsic coordinates

Note from figure 5 the arclength $\delta s$ is given by

$$\delta s = \rho \delta \psi + O(\delta \psi^2),$$

so that the radius of curvature $\rho = |ds/d\psi| = 1/|\kappa|$, and finally we find that

$$\boxed{\underline{\dot{v}} = \ddot{s}\underline{T} \pm \frac{\dot{s}^2}{\rho}\underline{N}.} \tag{6}$$

where the plus is taken when the curve has $d\psi/ds > 0$.

## Example 3: Application: Heavy bead moving on smooth convex wire

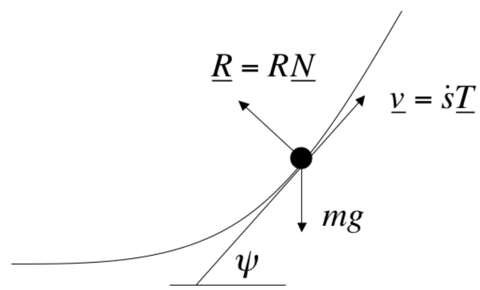


Figure 6: Heavy bead moving on smooth convex wire

We refer to figure 6. The bead moves down the wire under gravity. It is not moving in a straight line, so it is accelerating, and Newton's 1st law tells us that there must be a force causing this acceleration. The acceleration is given by equation (6) above. $\rho = +1/\kappa$ since the curve is convex.

We have $m\underline{\dot{v}} = \underline{F}$ where $\underline{F} = \underline{R} + mg(-\underline{j})$. Write $\underline{R} = R\underline{N}$ and resolve

$$\underline{j} = (\underline{T} \cdot \underline{j})\underline{T} + (\underline{N} \cdot \underline{j})\underline{N} = \underline{T}\sin\psi + \underline{N}\cos\psi$$

to obtain

$$m\left(\ddot{s}\underline{T} + \frac{\dot{s}^2}{\rho}\underline{N}\right) = m\underline{\dot{v}} = R\underline{N} - mg(\underline{T}\sin\psi + \underline{N}\cos\psi).$$

Hence, equating coefficients of the orthonormal pair $\underline{T}, \underline{N}$ we obtain:

$$m\ddot{s} = -mg\sin\psi \tag{7}$$

$$m\frac{\dot{s}^2}{\rho} = R - mg\cos\psi \tag{8}$$

These compare with the cartesian coordinate version:

$$\begin{aligned} m\ddot{x} &= R\cos\psi \\ m\ddot{y} &= -mg + R\sin\psi \end{aligned}$$

To illustrate, we'll try two different curves.

## Example 4: Heavy bead moving on cycloid given by $x = a(\theta + \sin\theta), y = a(1 - \cos\theta)$

Recall that $s = 4a\sin\psi$ is the equation for the cycloid in intrinsic coordinates. Thus from (7) we have $\ddot{s} = -\frac{gs}{4a}$. This is a simple harmonic oscillator with angular frequency $\omega = \sqrt{g/4a}$ and period $2\pi/\omega = 2\pi\sqrt{4a/g} = 4\pi\sqrt{a/g}$.

## Example 5: Heavy bead moving on the curve $y = -\log(\cos x)$ for $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$

We have $\tan\psi = dy/dx = \tan x$ and $ds/dx = \sqrt{1+\tan^2 x} = \sec x$ so $s = \int \sec x\, dx$. To do this integral use $t = \tan(x/2)$. Then

$$s = \int \sec x\, dx = \int \frac{1+t^2}{1-t^2} \times \frac{2}{1+t^2}\, dt = \int \frac{2}{1-t^2}\, dt = \int \frac{1}{1-t} + \frac{1}{1+t}\, dt = \log\left|\frac{1+t}{1-t}\right| + C.$$

With $x = 0$ when $s = 0$ we have $C = 0$. Thus

$$(1-t)e^s = 1 + t,$$

which rearranges to

$$t = \frac{e^s - 1}{e^s + 1} = \tanh\frac{s}{2}.$$

Thus, since $t = \tan\frac{x}{2}$ we obtain

$$x(s) = 2\tan^{-1}\left(\tanh\frac{s}{2}\right).$$

We have $\tan x = \tan\psi$ and so $x = \psi + k\pi$ and looking at the curve, $x = 0$ when $\psi = 0$, and so $k = 0$, i.e. $x = \pi$. Thus

$$\sin\psi = \sin x = \frac{2t}{1+t^2} = \frac{2\tanh\frac{s}{2}}{1+\tanh^2\frac{s}{2}} = \frac{2\sinh\frac{s}{2}\cosh\frac{s}{2}}{\cosh^2\frac{s}{2} + \sinh^2\frac{s}{2}} = \tanh s.$$

This gives

$$\ddot{s} = -g\sin\psi = -g\tanh s.$$

Now $\frac{d}{ds}\tanh s = \text{sech}^2 s$ so that by a Maclaurin series for $\tanh(s)$ about $s = 0$,

$$\tanh s = \tanh(0) + \frac{1}{1!}\text{sech}^2(0)s + O(s^2),$$

gives $\tanh s = s + O(s^2)$. Hence for small oscillations, the motion is simple harmonic:

$$\ddot{s} = -gs.$$

with period $2\pi/\sqrt{g}$.

## 3 Conservation of energy

We have shown that

$$m\ddot{s} = -mg\sin\psi.$$

Now (figure 2)

$$\sin\psi = \frac{dy}{ds},$$

so that,

$$m\dot{s}\ddot{s} = -mg\sin\psi\dot{s} = -mg\frac{dy}{ds}\frac{ds}{dt} = -mg\frac{dy}{dt},$$

and integrating yields

$$\frac{1}{2}m\dot{s}^2 + mgy = E,$$

where $E$ is a constant which recognise as the total conserved energy of the system. Here $\frac{1}{2}m\dot{s}^2$ is the kinetic energy and $mgy$ is the potential energy.

## Example 6: Heavy ball moving on outside of a smooth sphere

(Using intrinsic coordinates for this example is a bit of an overkill, since the accelerations are well-known for circular motion, but it helps as an illustration nevertheless. The acceleration/force of the ball keeping it in circular path is called centripetal.)

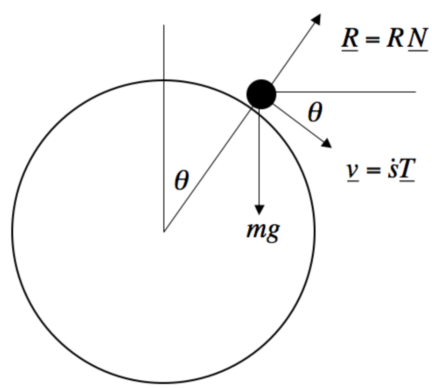


Figure 7: Ball moving on outside of a smooth sphere

Taking $s = 0$ at the top of the sphere, by geometry we have $s = a\theta$ and $\psi = 2\pi - \theta$. The normal points outwards and the curvature is $\kappa = -1/a$ (since $\psi$ decreases as $s$ increases). Thus, now we may use the equations (7) and (8) already derived to obtain

$$\ddot{s} = -g\sin\psi \quad (9)$$

$$m\frac{\dot{s}^2}{(-a)} = R - mg\cos\psi \quad (10)$$

simplifies to (using $s = a\theta$ and $\theta = 2\pi - \psi$)

$$\begin{aligned} a\ddot{\theta} &= g\sin\theta \\ m\frac{a^2\dot{\theta}^2}{a} &= mg\cos\theta - R, \end{aligned}$$

which is the more familiar set of equations obtained by using polar coordinates (see later lectures) From the first equation,

$$\dot{\theta}\ddot{\theta} = \frac{g}{a}\sin\theta\dot{\theta},$$

so that integrating w.r.t. $t$,

$$\frac{\dot{\theta}^2}{2} + \frac{g}{a}\cos\theta = E,$$

a constant (this is conservation of energy as already proved above). Using that $\theta = 0, \dot{\theta} = 0$ we have $E = g/a$ and hence

$$\frac{\dot{\theta}^2}{2} = \frac{g}{a}(1 - \cos\theta).$$

Now use the second equation to find that $R = mg\cos\theta - ma\dot{\theta}^2 = mg\cos\theta - 2mg(1-\cos\theta) = mg(3\cos\theta - 2)$. Thus, in particular, the ball leaves the sphere when $\theta = \cos^{-1}(2/3)$, since then $R = 0$.

## Example 7: Projectile motion in intrinsic coordinates

Consider a heavy particle moving in a medium that has a resistance proportional to (speed)$^n$ where $n$ is an integer. Find the position $x$ as a function of $\psi$. The particle is projected at speed $U$ and angle $\theta \in (0, \frac{\pi}{2})$ to the positive horizontal (see figure 8).

Note that $s$ is non-decreasing (the particle does not move just vertically) so $\dot{s}$ will be non-negative and $|\underline{v}| = \dot{s}$ is the speed of the particle.

From Newton's 2nd law,

$$m\dot{\underline{v}} = -k|\underline{v}|^n\underline{T} - mg\underline{j},$$

where $k > 0$ is a constant. Thus

$$\ddot{s}\underline{T} + \kappa\dot{s}^2\underline{N} = -\frac{k}{m}\dot{s}^n\underline{T} - g\underline{j}$$

Now $\underline{j} = \underline{T}\sin\psi + \underline{N}\cos\psi$. This gives

$$\ddot{s}\underline{T} + \kappa\dot{s}^2\underline{N} = -\frac{k}{m}\dot{s}^n\underline{T} - g(\underline{T}\sin\psi + \underline{N}\cos\psi)$$

Equating coefficients of $\underline{T}, \underline{N}$:

$$\begin{aligned} \ddot{s} &= -\frac{k}{m}\dot{s}^n - g\sin\psi \\ \kappa\dot{s}^2 &= -g\cos\psi. \end{aligned}$$

Let $v = \dot{s}$. Note that $v = \dot{s} = \frac{ds}{d\psi}\dot{\psi} = \kappa^{-1}\dot{\psi}$ so that the previous equations give

$$\begin{aligned} \dot{v} &= -\frac{k}{m}v^n - g\sin\psi \\ v\dot{\psi} &= -g\cos\psi. \end{aligned}$$

Now suppose that $n = 1$. Then these become

$$\begin{aligned} \dot{v} &= -\frac{k}{m}v - g\sin\psi \\ v\dot{\psi} &= -g\cos\psi. \end{aligned}$$

It turns out we can solve these (it is rare to be able to solve pairs of differential equations explicitly): divide out to get

$$\frac{1}{v}\frac{dv}{d\psi} = \tan\psi + \alpha v \sec\psi,$$

where $\alpha = \frac{k}{mg}$. This is a bit tricky to solve, unless you substitute $p = 1/v$ to obtain

$$p\left(-\frac{1}{p^2}\frac{dp}{d\psi}\right) = \tan\psi + \frac{\alpha\sec\psi}{p},$$

(you would not be expected to guess such a substitution in a question; you'd be told which substitution to make) which tidies up to the linear equation

$$\frac{dp}{d\psi} + p\tan\psi = -\alpha\sec\psi,$$

The integrating factor is $\exp\int\tan\psi\,d\psi = \sec\psi$:

$$\frac{d(p\sec\psi)}{d\psi} = -\alpha\sec^2\psi,$$

Thus

$$p\sec\psi = -\alpha\int\sec^2\psi\,d\psi = -\alpha(A + \tan\psi),$$

Where $A$ is a constant. Thus

$$v = -\frac{\sec\psi}{\alpha(A + \tan\psi)}.$$

We are told that initially $\psi = \theta$ and $v = U$. Thus we may find $A$:

$$A = -\tan\theta - \frac{\sec\theta}{\alpha U}.$$

To find $x$ in terms of $\psi$ we need a differential equation for $x$ in terms of $\psi$. Thus find $dx/d\psi$:

$$\frac{dx}{d\psi} = \frac{dx}{ds}\frac{ds}{d\psi} = \frac{dx}{ds}\frac{ds}{dt}\left(\frac{d\psi}{dt}\right)^{-1} = \frac{\cos\psi v}{\dot{\psi}}$$

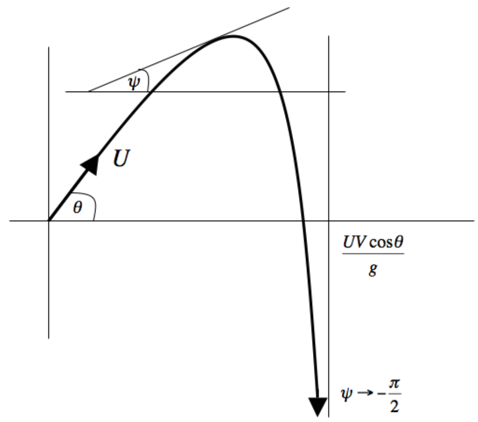


Figure 8: Particle moving in medium with resistance proportional to speed using intrinsic coordinates

But $v\dot{\psi} = -g\cos\psi$, so that we obtain (leaving $A$ as it is for now)

$$\frac{dx}{d\psi} = \frac{\cos\psi v}{\dot{\psi}} = -\frac{v^2}{g} = -\frac{1}{\alpha^2 g}\left(\frac{\sec\psi}{A+\tan\psi}\right)^2.$$

This shows that $\psi$ is non-increasing with $x$ (since $d\psi/dx \leq 0$). Now spot that

$$\frac{d}{d\psi}\left(\frac{1}{A+\tan\psi}\right) = -\frac{1}{(A+\tan\psi)^2} \times \sec^2\psi,$$

To see that

$$x(\psi) = B + \frac{1}{\alpha^2 g(A+\tan\psi)},$$

where $B$ is a constant. But $x=0$ when $\psi=\theta$, so that

$$0 = B + \frac{1}{\alpha^2 g(A+\tan\theta)} = B + \frac{1}{\alpha^2 g\left(-\frac{\sec\theta}{\alpha U}\right)},$$

giving $B = \dfrac{U}{\alpha g}\cos\theta$. Hence we have

$$x(\psi) = \frac{U\cos\theta}{\alpha g} + \frac{1}{\alpha^2 g\left(\tan\psi - \tan\theta - \left(\frac{\sec\theta}{\alpha U}\right)\right)}.$$

Now, one expects the particle to reach terminal velocity $V$ given by $V = gm/k = 1/\alpha$, so that we obtain

$$x(\psi) = \frac{UV\cos\theta}{g} + \frac{V^2\cos\theta}{g(\cos\theta\tan\psi - \sin\theta - \frac{V}{U})},$$

and finally

$$x(\psi) = \frac{UV\cos\theta}{g} + \frac{UV^2\cos\theta\cos\psi}{g(U\sin(\psi-\theta) - V\cos\psi)}.$$