MATH1302, Week 1

Vector dynamics and simple planar motion

1 Vector dynamics

We will be considering particles moving in 1,2 and 3 dimensions. In 2 and 3 dimensions vectors are used to describe particle position, velocity and acceleration.

Let a particle move along a curve; we can describe the position of the particle as a function of t as the position vector $\underline{r}$ (relative to a suitable origin O). Then we define

$$\begin{aligned} \textit{Velocity:} \quad \underline{v}(t) &= \frac{d\underline{r}}{dt} \\ &= \lim_{\tau \to 0} \frac{1}{\tau}(\underline{r}(t+\tau) - \underline{r}(t)). \\ \textit{Acceleration:} \quad \underline{a}(t) &= \frac{d\underline{v}}{dt} = \frac{d^2\underline{r}}{dt}. \end{aligned}$$

These are vector quantities: they have magnitude and direction. The magnitude $|\underline{v}|$ of the velocity is called the *speed*. By definition it is non-negative. Thus a particle thrown vertically upwards with velocity $\underline{U}$ and speed $U = |\underline{U}|$ (see figure 2) returns to the thrower with speed U (prove this!), but its velocity has then changed to $-\underline{U}$. Notice that the set of basis vectors

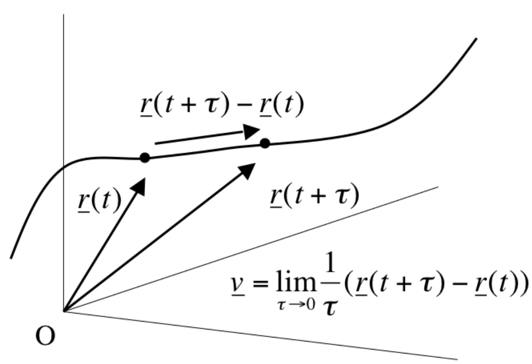

Figure 1: Definition of velocity

used to describe $\underline{r}$ does not come into the above definition. Note also that the velocity is independent of the choice of origin. When $\underline{r}$ is written in terms of the standard basis $\underline{i}, \underline{j}, \underline{k}$ we

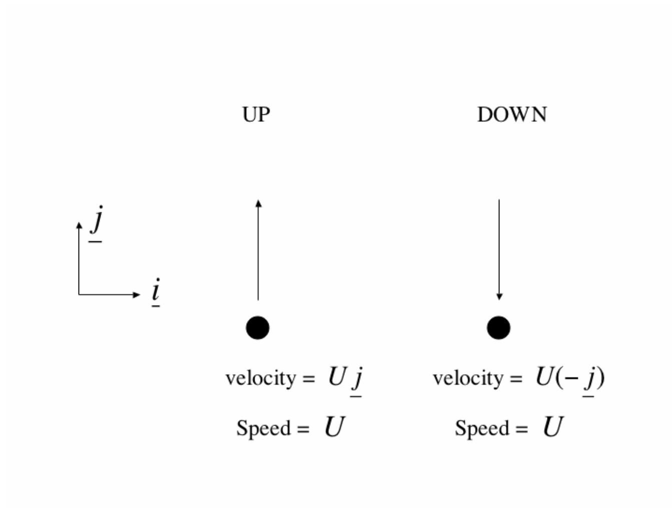

Figure 2: When a ball thrown upwards returns, its speed is the same, but its velocity has changed sign.

have $\underline{r} = x\underline{i} + y\underline{j} + z\underline{j}$ and hence

$$\begin{aligned}
\underline{v}(t) &= \frac{d\underline{r}}{dt} \\
&= \lim_{\tau \to 0} \frac{1}{\tau}(\underline{r}(t+\tau) - \underline{r}(t)) \\
&= \lim_{\tau \to 0} \frac{1}{\tau}(x(t+\tau) - x(t))\underline{i} \\
&\quad + \lim_{\tau \to 0} \frac{1}{\tau}(y(t+\tau) - y(t))\underline{j} + \lim_{\tau \to 0} \frac{1}{\tau}(z(t+\tau) - z(t))\underline{k} \\
&= \dot{x}\underline{i} + \dot{y}\underline{j} + \dot{z}\underline{k}
\end{aligned}$$

The speed $|\underline{v}| = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$.

Note: The lectures will start with solving some mechanics problems in vector notation. The problems can be solved without vector notation, by resolving force and accelerations vertically, horizontally, parallel to a plane, etc. The idea is to achieve some familiarity with vector mechanics before moving on to more difficult problems where the motion is on a curved plane, wire or surface, and where the basis vectors we work with change with position.

Example 1: Simple harmonic motion

A particle mass m moves in one dimension under a linear restoring force. Thus $\underline{r} = x\underline{i}$ and $\underline{v} = \dot{x}\underline{i}$, $\underline{a} = \ddot{x}\underline{i}$. The force $\underline{F} = mkx(-\underline{i})$ and so $m\underline{a} = \underline{F}$ (see later) gives $m\ddot{x}\underline{i} = mkx(-\underline{i})$ i.e.

$$\ddot{x} = -kx.$$

This has general solution $x = A\cos(\sqrt{k}t + \delta)$ where A, δ are constants. Hence

$$\begin{aligned} \underline{r} &= A\cos(\sqrt{k}t + \delta)\underline{i} \\ \underline{v} &= -A\sqrt{k}\sin(\sqrt{k}t + \delta)\underline{i} \\ \underline{a} &= -Ak\cos(\sqrt{k}t + \delta)\underline{i}. \end{aligned}$$

Note that the speed is NOT $\dot{x}$ but $|\dot{x}|$!

Example 2: Ball sliding down an inclined plane

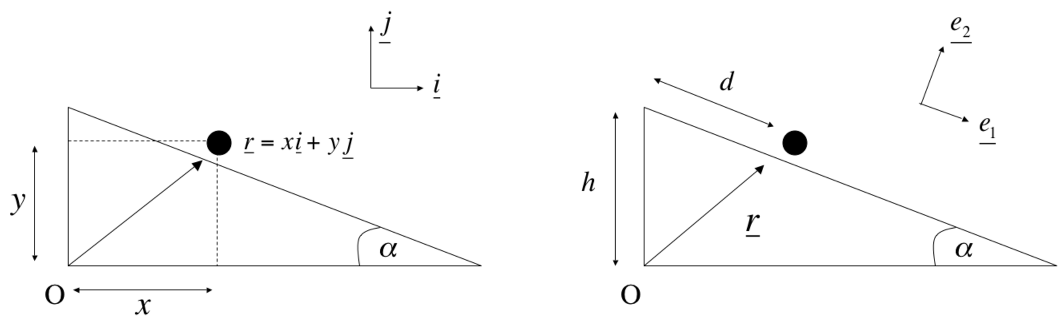

Figure 3: Ball rolling on inclined plane in standard coordinates (left) and in coordinates that use distance d travelled down the plane (right).

First using $\underline{i}, \underline{j}$ (left figure 3) we have $\underline{r} = x\underline{i} + y\underline{j}$ for the position vector of the particle. The velocity of the particle is thus $\underline{v} = \dot{x}\underline{i} + \dot{y}\underline{j}$.

On the other hand, in terms of the basis vectors $\underline{e_1}, \underline{e_2}$ (see right figure 3) we have

$$\underline{r} = \underline{h} + d\underline{e_1},$$

where $\underline{h}$ is the vector stretching from O to the top of the plane. Thus the velocity is

$$\frac{d}{dt}\underline{r} = \frac{d}{dt}(\underline{h} + d\underline{e_1}) = \dot{d}\underline{e_1}.$$

It is easy to show that the two velocities are equal: We note that $\underline{e_1}, \underline{e_2}$ are just $\underline{i}, \underline{j}$ rotated by angle α, so that

$$\underline{e_1} = \cos\alpha\underline{i} - \sin\alpha\underline{j}, \quad \underline{e_2} = \sin\alpha\underline{i} + \cos\alpha\underline{j}.$$

Moreover, $\cos\alpha = x/d$ so that $\dot{d}\cos\alpha = \dot{x}$. Hence

$$\dot{d}\underline{e_1} = \frac{\dot{x}}{\cos\alpha}\underline{e_1} = \frac{\dot{x}}{\cos\alpha}(\cos\alpha\underline{i} - \sin\alpha\underline{j}) = \dot{x}\underline{i} - \dot{x}\tan\alpha\underline{j} \qquad (1)$$

But $\tan\alpha = \dfrac{h-y}{x}$ which gives $x\tan\alpha = h - y$ and thus $\dot{x}\tan\alpha = -\dot{y}$, so that (1) becomes

$$\dot{d}\underline{e_1} = \dot{x}\underline{i} - \dot{x}\tan\alpha\underline{j} = \dot{x}\underline{i} + \dot{y}\underline{j},$$

which confirms that the velocity is the same whether we work with $\underline{i}, \underline{j}$ or $\underline{e_1}, \underline{e_2}$.

2 Newton's Laws

We assume, unless stated otherwise, that our frame of reference is at rest. To recap on MATH1301, these laws are

> **Newton's Laws of motion**
>
> **First Law** Every particle persists in its state of rest or uniform motion in a straight line unless it is compelled to change that state by impressed forces;
>
> **Second Law** The rate of change of motion is proportional to the motive force impressed; and it is made in the direction of the straight line through which the force is impressed;
>
> **Third Law** To every action there is an equal and opposite reaction: the mutual interactions of two bodies on each other is equal but oppositely directed.

The second law, which is just a quantitative statement of the first law, is usually written in equation form as

$$\text{force} = \text{mass} \times \text{acceleration},$$

or if $\underline{F}$ is the force acting on the particle of mass m,

$$\boxed{\underline{F} = m\underline{a}.}$$

Example 3: Particle moving vertically under gravity

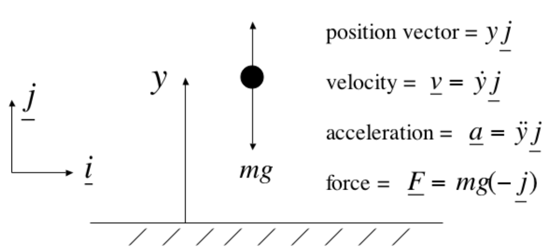

Figure 4: Ball moving vertically under gravity

Referring to figure 4, a ball is thrown upwards with initial speed U. Let $y(t)$ be the height of the ball at time t. Then Newton's 2nd law gives

$$\underline{F} = m\underline{a} \;\Rightarrow\; mg(-\underline{j}) = m \times \ddot{y}\underline{j}.$$

Hence we have $\ddot{y} = -g$. Integrating once $\dot{y} = -gt + C$, C constant. But at $t = 0$, $\dot{y} = U$, so that $U = -g0 + C$, i.e. $C = U$ which gives $\dot{y} = U - gt$. Let's check this: As t grows from zero, $\dot{y}$ reduces from U to zero at $t = U/g$, i.e. at the highest point the ball reaches, after which $\dot{y}$ becomes negative, that is y starts to decrease. This all agrees with what we expect. To find the height integrate again $y = Ut - \frac{1}{2}gt^2 + C'$, where C' is a constant. Now use that $y = 0$ at $t = 0$ to obtain $C' = 0$ and hence $y = Ut - \frac{1}{2}gt^2$.

Example 4: Ball sliding down an inclined plane

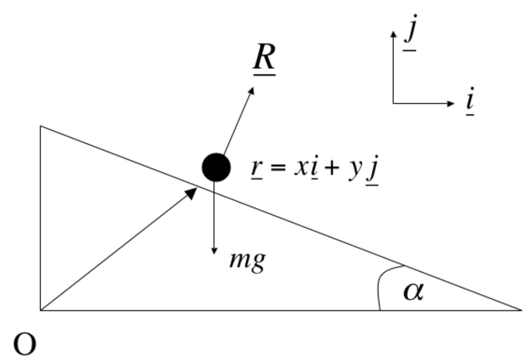

Figure 5: Ball sliding down an inclined plane

Refer to figure 5. We have

$$\text{total force on ball } = \underline{R} + mg(-\underline{j}).$$

The acceleration $\underline{a} = \ddot{x}\underline{i} + \ddot{y}\underline{j}$. Hence Newton's 2nd law gives

$$\underline{R} + mg(-\underline{j}) = m(\ddot{x}\underline{i} + \ddot{y}\underline{j}).$$

Now write $\underline{R} = R_1\underline{i} + R_2\underline{j}$ so that $R_1 = \underline{R} \cdot \underline{i} = R\sin\alpha$ and $R_2 = \underline{R} \cdot \underline{j} = R\cos\alpha$

$$R(\sin\alpha\underline{i} + \cos\alpha\underline{j}) + mg(-\underline{j}) = m(\ddot{x}\underline{i} + \ddot{y}\underline{j}).$$

Now equation coefficients (of the linearly independent) vectors $\underline{i}, \underline{j}$ we obtain the two scalar equations

$$\begin{aligned} m\ddot{x} &= R\sin\alpha \\ m\ddot{y} &= R\cos\alpha - mg \end{aligned}$$

Now eliminate the R:

$$m \sin \alpha \ddot{y} = R \sin \alpha \cos \alpha - mg \sin \alpha = m \cos \alpha \ddot{x} - mg \sin \alpha.$$

This gives $\ddot{y} = \ddot{x} \cot \alpha - g$. Integrate once:

$$\dot{y} = \dot{x} \cot \alpha - gt + C.$$

The constant C is found by using that $\dot{x} = 0$ and $\dot{y} = 0$ at $t = 0$, i.e. $C = 0$, which gives $\dot{y} = \dot{x} \cot \alpha - gt$. Now integrate a second time: $y = x \cot \alpha - \frac{1}{2} g t^2 + C'$. Now use that at $t = 0$, $x = 0$ and $y = h$ to obtain $C' = h$ and hence

$$y = h + x \cot \alpha - \frac{1}{2} g t^2.$$

Finally by the geometry, $\tan \alpha = \frac{h-y}{x}$ and so eliminating x we have (after some algebra)

$$y = h - \frac{1}{2} g \sin^2 \alpha t^2,$$

and

$$x = \frac{1}{2} g \sin \alpha \cos \alpha t^2.$$

Example 5: Two slabs on a smooth hill connected by a light inextensible string

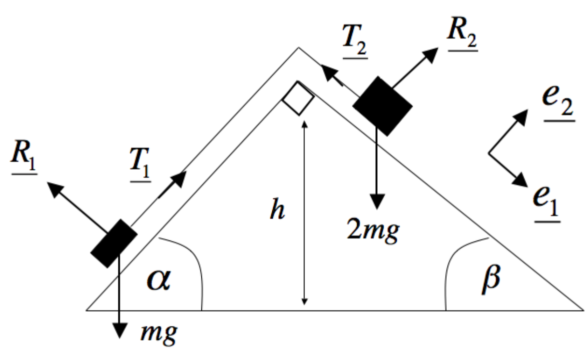

Figure 6: Two heavy slabs on a smooth hill connected by a light inextensible string.

See figure 6 where $\alpha + \beta = \dfrac{\pi}{2}$ so that the top angle is a rightangle (e.g. $\alpha = \beta = \pi/4$). Two heavy slabs are connected by a light inextensible string. Let d be the distance that the slab of mass m has moved up the hill. We do not need to worry about the normal reaction at the hill top as the only forces that influence the block subsystems are the tension (which must be the same in magnitude in each part of the string as there is no friction), the weights and the reaction forces of the slabs on the hill. We have $\underline{T_1} = T\underline{e_2}$ and $\underline{T_2} = T(-\underline{e_1})$. Also $\underline{R_1} = R_1(-\underline{e_1})$ and $\underline{R_2} = R_2\underline{e_2}$. This gives

$$\begin{aligned} 2m\ddot{d}\underline{e_1} &= -T\underline{e_1} + R_2\underline{e_2} + 2mg\sin\beta\underline{e_1} - 2mg\cos\beta\underline{e_2} \\ m\ddot{d}\underline{e_2} &= T\underline{e_2} - mg\sin\alpha\underline{e_2} + mg\cos\alpha\underline{e_1} - R_1\underline{e_1}. \end{aligned}$$

Equating coefficients of $\underline{e_1}, \underline{e_2}$ and using that $\sin\alpha = \cos\beta$ and $\cos\alpha = \sin\beta$ gives

$$\begin{aligned} 2mg\cos\alpha - T &= 2m\ddot{d} \\ T - mg\sin\alpha &= m\ddot{d} \\ R_2 &= 2mg\sin\alpha \\ R_1 &= mg\cos\alpha. \end{aligned}$$

Adding the first two equations: $3m\ddot{d} = 2mg\cos\alpha + mg\sin\alpha$ so that since the slabs start at rest,

$$d(t) = \frac{gt^2}{6}(2\cos\alpha + \sin\alpha).$$

Example 6: Particle projected in vertical moving under gravity

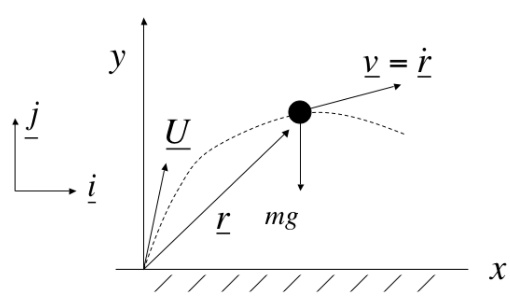

Figure 7: Particle projected in vertical plane under gravity

See figure 7. Take $\underline{i}, \underline{j}$ as unit vectors in the x, y directions. Then the position of the particle (say mass m) at time t is $\underline{r}(t) = x(t)\underline{i} + y(t)\underline{j}$. Let $\underline{v} = \frac{d}{dt}\underline{r}$ be its velocity. Then Newton's law gives

$$\underline{F} = m\underline{a} = m\dot{\underline{v}},$$

where $\underline{F} = mg(-\underline{j})$ is the total force acting on the particle. Thus

$$\dot{\underline{v}} = -g\underline{j}, \tag{2}$$

This is a vector differential equation, and may be treated just like a scalar differential equation, except we must add on an arbitrary constant vector. Thus integrating (2) we have

$$\int \dot{\underline{v}} dt = \int -g\underline{j}\, dt$$

so $\underline{v} = \underline{C} - gt\underline{j}$. For $\underline{C}$ use $t = 0, \underline{v} = \underline{U}$ if the particle has initial velocity $\underline{U}$: get $\underline{C} = \underline{U}$. Hence $\underline{v} = \underline{U} - gt\underline{j}$. The position $\underline{r}$ is obtained by integrating again:

$$\underline{r} = \int \underline{v}\, dt = \int \underline{U} - gt\underline{j}\, dt = \underline{U}t - \frac{1}{2}gt^2\underline{j} + \underline{C}'.$$

If initially $\underline{r} = \underline{r}_0$ then $\underline{C}' = \underline{r}_0$ and so

$$\underline{r} = \underline{r}_0 + \underline{U}t - \frac{1}{2}gt^2\underline{j}.$$

If particle is projected from the origin ($\underline{r}_0 = 0$) with speed U at an angle α to the horizontal then $\underline{U} = U\cos\alpha\underline{i} + U\sin\alpha\underline{j}$ and hence $\underline{r} = U\cos\alpha t\underline{i} + U\sin\alpha t\underline{j} - (gt^2/2)\underline{j}$. Equating components of $\underline{i}, \underline{j}$ gives

$$\begin{aligned} x &= U\cos\alpha t \\ y &= U\sin\alpha t - \frac{1}{2}gt^2. \end{aligned}$$

So particle moves with constant speed in the horizontal (there is no horizontal component of the force) and under constant acceleration in the vertical (under the constant gravitational force).

Horizontal Range

To find horizontal range of the particle, we need to find T when $y(T) = 0$. This is T such that $0 = y = U\sin\alpha T - gT^2/2$. Thus either $T = 0$ or $T = 2U\sin\alpha/g$. $T = 0$ is where the particle starts, so we need $T = 2U\sin\alpha/g$ and hence the range

$$\boxed{x_{\max} = x(T) = \frac{U^2}{g}\sin(2\alpha).}$$

Cartesian equation for the path of the particle ($\alpha \neq \pi/2$)

For the path of the particle, we must eliminate t to find y as a function of x. Thus we have $t = \frac{x}{U\cos\alpha}$ $(\alpha \neq \pi/2)$ and hence $y = U\sin\alpha t - gt^2/2 = U\sin\alpha\frac{x}{U\cos\alpha} - \frac{g}{2}(\frac{x}{U\cos\alpha})^2 = x\tan\alpha - \frac{gx^2}{2U^2}\sec^2\alpha$. So that

$$\boxed{y = x\tan\alpha - \frac{gx^2}{2U^2}(1 + \tan^2\alpha) \;\; (\alpha \neq \frac{\pi}{2}).} \tag{3}$$

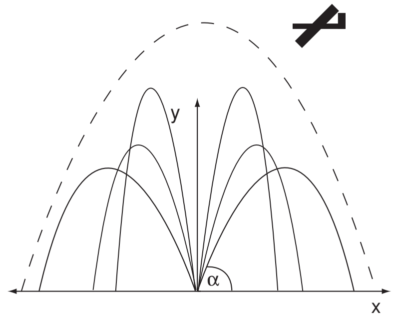

Figure 8: Parabola of safety (the dotted line in figure)

Parabola/paraboloid of safety

Suppose that an anti-aircraft gun is placed at the origin and it fires shells at a fixed speed U, but at any angle α. If an airplane wishes to fly over the gun, what is the shape of the area of space that it must avoid? If the position of the airplane is X, Y $(X \neq 0)$, then we would like to find condition(s) that the shell cannot reach the point (X, Y) no matter what angle α is used. Thus from the previous paragraph we want (X, Y) such that the equation (3) at X, Y :

$$Y = X \tan\alpha - \frac{gX^2}{2U^2}(1 + \tan^2\alpha)$$

has no real solutions in α, or equivalently $\tan\alpha$. Write as a quadratic in $\tan\alpha$: we get

$$\frac{gX^2}{2U^2}\tan^2\alpha - X\tan\alpha + \left(Y + \frac{gX^2}{2U^2}\right) = 0.$$

This has no real solutions in $\tan\alpha$ if

$$X^2 < 4\left(\frac{gX^2}{2U^2}\right)\left(Y + \frac{gX^2}{2U^2}\right).$$

Hence plane is safe if $Y > \frac{U^2}{2g} - \frac{gX^2}{2U^2}$ when $X \neq 0$. By continuity it is also safe when $X = 0$. Thus the airplane is safe if it lies above the parabola

$$Y = \frac{U^2}{2g} - \frac{gX^2}{2U^2}.$$

By symmetry we can think of an air plane flying over a gun placed at the origin of the (x, y)-plane. If $r = \sqrt{x^2 + y^2}$ and z is the height of the airplane, the airplane is safe if it flies above the paraboloid

$$\boxed{z = \frac{U^2}{2g} - \frac{g(x^2 + y^2)}{2U^2}.}$$

Particle projected in vertical moving under gravity and air resistance

Suppose we are told that the air resistance on the particle is $k\underline{v}$ per unit mass. Now the total force on the particle is $\underline{F} = mg(-\underline{j}) + mk(-\underline{v})$. The minus before the $\underline{v}$ is because the frictional force acts in the *opposite* direction to the velocity. Thus from Newton's 2nd law we obtain

$$\dot{\underline{v}} = -g\underline{j} - k\underline{v}.$$

We may rewrite this as $\dot{\underline{v}} + k\underline{v} = -g\underline{j}$. If $\underline{v}$ were a scalar v we would use an integrating factor $\exp(\int k dt) = e^{kt}$. On reflection one sees that the same integrating factor works for the vector version:

$$\frac{d}{dt}(\underline{v}e^{kt}) = e^{kt}\dot{\underline{v}} + k\underline{v}e^{kt} = e^{kt}(\dot{\underline{v}} + k\underline{v}) = -g\underline{j}e^{kt}.$$

Integrating we obtain

$$\underline{v}e^{kt} = -\frac{g}{k}e^{kt}\underline{j} + \underline{C}.$$

Now use $\underline{v}(0) = \underline{U}$ to obtain

$$\underline{C} = \underline{U} + \frac{g}{k}\underline{j}$$

and tidy up to obtain

$$\underline{v}(t) = \left(\frac{g}{k}\underline{j} + \underline{U}\right) e^{-kt} - \frac{g}{k}\underline{j}.$$

Note that as $t \to \infty$ we have $\underline{v}(t) \to -\frac{g}{k}\underline{j}$. Thus the speed $|\underline{v}|$ of the particle tends to the terminal velocity g/k. Particle eventually moves vertically at terminal velocity ($\dot{x}(t) \to 0$, $\dot{y}(t) \to -g/k$). [check: y is measured upwards, so it is decreasing, and hence we expect the minus sign in $-g/k$ since then $dy/dt < 0$].

To find $\underline{r}(t)$ we integrate

$$\underline{v} = \frac{d\underline{r}}{dt} = \underline{v}(t) = \left(\frac{g}{k}\underline{j} + \underline{U}\right) e^{-kt} - \frac{g}{k}\underline{j}.$$

Thus

$$\underline{r} = \int \left(\frac{g}{k}\underline{j} + \underline{U}\right) e^{-kt} - \frac{g}{k}\underline{j}\, dt = \frac{-1}{k}\left(\frac{g}{k}\underline{j} + \underline{U}\right) e^{-kt} - \frac{g}{k}t\underline{j} + \underline{C}'.$$

Hence since $\underline{r}(0) = 0$ we get $\underline{C}' = \frac{1}{k}\left(\frac{g}{k}\underline{j} + \underline{U}\right)$ and hence

$$\underline{r}(t) = \left(\frac{g}{k^2}\underline{j} + \frac{\underline{U}}{k}\right)(1 - e^{-kt}) - \frac{g}{k}t\underline{j}.$$

using that $\underline{u} = U\cos\alpha\underline{i} + U\sin\alpha\underline{j}$ we obtain

$$\begin{aligned} x(t) &= \frac{U\cos\alpha}{k}(1 - e^{-kt}) \\ y(t) &= \left(\frac{g}{k^2} + \frac{U\sin\alpha}{k}\right)(1 - e^{-kt}) - \frac{gt}{k}. \end{aligned}$$

To find the particle path (y as a function of x, assuming $\alpha \neq \frac{\pi}{2}$), we need to eliminate t. It is easiest to find t as a function of x: since $x(t) = \frac{U\cos\alpha}{k}(1 - e^{-kt})$ we find

$$1 - e^{-kt} = \frac{kx}{U\cos\alpha}$$

and

$$t = \frac{1}{k}\log\left(\frac{U\cos\alpha}{U\cos\alpha - kx}\right).$$

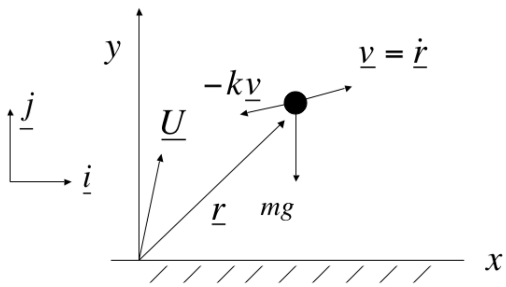

Figure 9: Particle projected in vertical plane under gravity and under air resistance proportional to speed

Now substiutute for t in the equation for y:

$$y = \left(\frac{g}{k^2} + \frac{U\sin\alpha}{k}\right)(1 - e^{-kt}) - \frac{g}{k}t = \left(\frac{g}{k^2} + \frac{U\sin\alpha}{k}\right)\left(\frac{kx}{U\cos\alpha}\right) - \frac{g}{k}\frac{1}{k}\log\left(\frac{U\cos\alpha}{U\cos\alpha - kx}\right).$$

Tidying up gives finally the cartesian equation for the particle path:

$$\boxed{y = \left(\frac{g}{kU\cos\alpha} + \tan\alpha\right)x + \frac{g}{k^2}\log\left(1 - \frac{kx}{U\cos\alpha}\right), \quad (\alpha \neq \frac{\pi}{2}).}$$

[Note that since $x(t) = \frac{U\cos\alpha}{k}(1 - e^{-kt})$, $1 - \frac{kx}{U\cos\alpha} = e^{-kt} > 0$ so the log makes sense.]