MATHEMATICS 1302 (Applied Mathematics 2)
YEAR 2008–2009, TERM 2

PRACTICE FINAL EXAM

Note: The actual exam will have 6 questions, of which the best 4 will count. Here I am giving you just 5 questions.

1. A body is projected vertically upwards from the surface of the earth with initial velocity v_0. Neglect air resistance, but do *not* make the approximation that g is constant.

   (a) Describe qualitatively the possible motions in as much detail as possible (e.g. explaining the asymptotic behavior as $t \to +\infty$). You will need to distinguish three cases: $v_0 < v_{escape}$, $v_0 = v_{escape}$ and $v_0 > v_{escape}$. Compute v_{escape} in terms of G, M_{earth} and R_{earth}.

   (b) Find the distance from the center of the earth as a function of time for the case $v_0 = v_{escape}$.

2. (a) Write the unit vectors $\widehat{\mathbf{e}}_r$ and $\widehat{\mathbf{e}}_\theta$ of plane polar coordinates in terms of the Cartesian unit vectors $\widehat{\mathbf{e}}_x$ and $\widehat{\mathbf{e}}_y$ and the coordinates r and θ.

   (b) Consider a particle moving with polar coordinates $(r(t), \theta(t))$. Using your result from part (a) — or by any other valid method — show that

$$\frac{d}{dt}\,\widehat{\mathbf{e}}_r = \dot{\theta}\,\widehat{\mathbf{e}}_\theta$$
$$\frac{d}{dt}\,\widehat{\mathbf{e}}_\theta = -\dot{\theta}\,\widehat{\mathbf{e}}_r$$

   where $\dot{\theta}$ denotes $d\theta/dt$.

   (c) Write the position vector $\mathbf{r}$, the velocity vector $\mathbf{v} = d\mathbf{r}/dt$, the acceleration vector $\mathbf{a} = d^2\mathbf{r}/dt^2$ and the "jerk" vector $\mathbf{j} = d^3\mathbf{r}/dt^3$ as linear combinations of $\widehat{\mathbf{e}}_r$ and $\widehat{\mathbf{e}}_\theta$ with coefficients that involve r, θ and their time derivatives.

3. A particle slides frictionlessly under the influence of gravity on the inverted parabola $y = -\dfrac{x^2}{2a}$.

   (a) Write the horizontal (x) and vertical (y) components of Newton's equations of motion.

   (b) Eliminate the constraint and the constraint force to get a differential equation for $x(t)$ alone.

(c) Write the equation of energy conservation for the motion $x(t)$.

(d) Differentiate the energy-conservation equation and show that it agrees with the equation of motion derived in part (b).

(e) Write $t(x)$ in the form of a definite integral, being careful about initial conditions. But don't bother to evaluate the integral.

(f) Assume that the particle starts from $x = 0$ with initial velocity v_0. Show that there exist three values of v_0 for which the subsequent motion is $x(t) = v_0 t$, and find those values.

4. A simple pendulum of length ℓ, whose bob has a mass m, is attached to a support moving vertically upward with constant acceleration a. Let θ be the angle of the pendulum relative to the vertical.

   (a) Write the Cartesian coordinates $x(t)$ and $y(t)$ of the pendulum bob in terms of the angle $\theta(t)$.

   (b) Write the horizontal (x) and vertical (y) components of Newton's equations of motion.

   (c) Eliminate the constraint and the constraint force to get a differential equation for $\theta(t)$ alone.

   (d) Find the frequency of small oscillations around $\theta = 0$.

   (e) Is energy conserved? Why or why not?

5. A smooth thin wire is bent into the shape $z = f(x)$, where f is some specified function satisfying $f(x) = f(-x)$. This wire is then made to rotate with angular velocity ω about the z axis [i.e. about the point $x = 0$ on the wire]; here the $+z$ direction is of course oriented upwards. A bead of mass m then slides frictionlessly on the wire under the influence of gravity. Using cylindrical coordinates (ρ, φ, z), discuss the motion of the bead, as follows. Write your answers in terms of the function f and its derivatives.

   (a) Write the ρ, φ and z components of Newton's equations of motion for the bead. Your equations will contain two unknown constraint forces.

   (b) Use the equations of constraint to eliminate all reference to φ, z, and their time derivatives as well as to the constraint forces. That is, you should obtain a differential equation for ρ alone.

   (c) Fix some value ρ_0. What value of ω will allow the bead to remain at rest at $\rho = \rho_0$?

   (d) If ω is chosen as in part (d), under what conditions is the solution $\rho = \rho_0$ stable? When it is stable, find the frequency of small oscillations about the solution $\rho = \rho_0$.