MATHEMATICS 1302 (Applied Mathematics 2)
YEAR 2008–2009, TERM 2

HANDOUT #8: COUPLED OSCILLATIONS AND NORMAL MODES

1 Review of simple harmonic oscillator

In MATH 1301 you studied the *simple harmonic oscillator*: this is the name given to any physical system (be it mechanical, electrical or some other kind) with one degree of freedom (i.e. one dependent variable x) satisfying the equation of motion

$$m\ddot{x} \;=\; -kx \;, \tag{1}$$

where m and k are constants (and the dot $\dot{}$ denotes d/dt as usual). For instance, if we have a particle of mass m attached to a spring of spring constant k (with the other end of the spring attached to a fixed wall), then the force on the particle is $F = -kx$ where x is the particle's position (with $x = 0$ taken to be the equilibrium point of the spring), so Newton's Second Law $F = ma$ is indeed (1).

Let us review briefly the solution of the harmonic-oscillator equation (1). Since this is a one-dimensional problem with a position-dependent force, it can be solved by the energy method, with potential energy $U(x) = \frac{1}{2}kx^2$. [This was done as a worked example in one of the Kleppner & Kolenkow handouts.] But a simpler method is to recognize that (1) is a homogeneous linear differential equation with constant coefficients, so its solutions can be written (except in certain degenerate cases) as linear combinations of suitably chosen exponentials, which we can write either as $x(t) = e^{\alpha t}$ or as $x(t) = e^{i\omega t}$. Let us use the latter form (which is more convenient for oscillatory systems, because ω will come out to be a real number). So the method is to *guess* a solution of the form

$$x(t) \;=\; e^{i\omega t} \tag{2}$$

and then choose ω so that this indeed solves (1). Inserting $x(t) = e^{i\omega t}$ into (1), we find

$$-m\omega^2 e^{i\omega t} \;=\; -ke^{i\omega t} \;, \tag{3}$$

which is a solution if (and only if) $\omega = \pm\sqrt{k/m}$. We conclude that the general solution of (1) is

$$x(t) \;=\; Ae^{i\omega t} \,+\, Be^{-i\omega t} \tag{4}$$

with $\omega = \sqrt{k/m}$.[1] This can equivalently be written in "sine-cosine" form as

$$x(t) \;=\; C_1 \cos\omega t \,+\, C_2 \sin\omega t \tag{5}$$

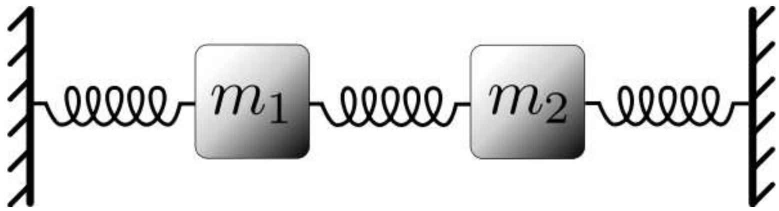
or in "amplitude-phase" form as

$$x(t) \;=\; C\cos(\omega t + \phi) \;. \tag{6}$$

[1]This is wrong when $\omega = 0$, because then the solutions $e^{i\omega t}$ and $e^{-i\omega t}$ are not linearly independent. In this case the linearly independent solutions are $e^{i\omega t}$ (i.e. 1) and $te^{i\omega t}$ (i.e. t). This is the "degenerate case" I referred to earlier.

2 Coupled oscillations: A simple example

Now let us consider a simple situation with *two* degrees of freedom. Suppose we have two particles, of masses m_1 and m_2, respectively, connected as follows:



Let us assume for simplicity that all three springs have the same spring constant k, and that the distance between the walls is exactly the sum of the equilibrium lengths of the three springs. Then the equations of motion are

$$m_1\ddot{x}_1 = -kx_1 + k(x_2 - x_1) \tag{7a}$$

$$m_2\ddot{x}_2 = -kx_2 + k(x_1 - x_2) \tag{7b}$$

(You should check this carefully and make sure you understand the *signs* on all four forces.)

Let us now try a solution of the form

$$x_1(t) = A_1 e^{i\omega t} \tag{8a}$$

$$x_2(t) = A_2 e^{i\omega t} \tag{8b}$$

where A_1 and A_2 are constants. Substituting this into (7) yields

$$-m_1\omega^2 A_1 e^{i\omega t} = -2kA_1 e^{i\omega t} + kA_2 e^{i\omega t} \tag{9a}$$

$$-m_2\omega^2 A_2 e^{i\omega t} = kA_1 e^{i\omega t} - 2kA_2 e^{i\omega t} \tag{9b}$$

Extracting the common factor $e^{i\omega t}$ and moving everything to the right-hand side, we obtain

$$(m_1\omega^2 - 2k)A_1 + kA_2 = 0 \tag{10a}$$

$$kA_1 + (m_2\omega^2 - 2k)A_2 = 0 \tag{10b}$$

which is most conveniently written in matrix form as

$$\begin{pmatrix} m_1\omega^2 - 2k & k \\ k & m_2\omega^2 - 2k \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} . \tag{11}$$

This is a homogeneous linear equation; it has a nonzero solution (i.e. a solution other than $A_1 = A_2 = 0$) if and only if the matrix on the left-hand side is singular, i.e. has a zero determinant. Setting the determinant equal to zero gives a quadratic equation for ω^2, namely

$$(m_1\omega^2 - 2k)(m_2\omega^2 - 2k) - k^2 = 0 , \tag{12}$$

which can be solved by the quadratic formula. Then, for each of the two possible values for ω^2, we can go back to the linear equation (11) and solve for the "eigenvector" $\begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$.

In particular, in the "symmetric case" $m_1 = m_2 = m$, the solutions are

$$\omega_1 = \sqrt{k/m} \tag{13a}$$

$$\omega_2 = \sqrt{3k/m} \tag{13b}$$

and the corresponding eigenvectors are

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{14a}$$

$$\mathbf{e}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{14b}$$

The **normal modes** — that is, the solutions of (7) that are pure oscillations at a single frequency — are therefore

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\sqrt{k/m}\, t + \phi_1) \tag{15}$$

and

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(\sqrt{3k/m}\, t + \phi_2)\ . \tag{16}$$

The general solution is a linear combination of these two normal modes. In the first (slower) normal mode, the two particles are oscillating in phase, with the same amplitude; the middle spring therefore exerts no force at all, and the frequency is $\sqrt{k/m}$ as it would be if the middle spring were simply absent. In the second (faster) normal mode, the two particles are oscillating 180° out of phase, with the same amplitude; therefore, each particle feels a force that is $-3k$ times its displacement (why?), and the frequency is $\sqrt{3k/m}$.

This solution can be interpreted in another way. Let us build a matrix N whose columns are the eigenvectors corresponding to the normal modes,

$$N = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\ , \tag{17}$$

and let us make the change of variables

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = N \begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix} \tag{18}$$

or equivalently

$$x_1 = x'_1 + x'_2 \tag{19a}$$

$$x_2 = x'_1 - x'_2 \tag{19b}$$

Then a solution with $x'_1 \neq 0$, $x'_2 = 0$ corresponds to the first normal mode, while a solution with $x'_1 = 0$, $x'_2 \neq 0$ corresponds to the second normal mode. The point is that the change of variables (18)/(19) *decouples* the system (7) [when $m_1 = m_2 = m$]: after a bit of algebra we obtain

$$m\ddot{x}_1' = -kx_1' \tag{20a}$$

$$m\ddot{x}_2' = -3kx_2' \tag{20b}$$

(You should check this!) In other words, by a linear change of variables corresponding to passage to the normal modes, the system (7) of *coupled* harmonic oscillators turns into a system (20) of *decoupled* simple harmonic oscillators, each of which may be solved *separately* by the elementary method reviewed in Section 1.

3 Coupled oscillations: The general case

We can now see how to handle the general case of coupled oscillators with an arbitrary finite number n of degrees of freedom. We will have a system of homogeneous linear constant-coefficient differential equations of the form

$$M\ddot{\mathbf{x}} + K\mathbf{x} = \mathbf{0} \tag{21}$$

where

- $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ is a column vector of coordinates;
- M (the so-called **mass matrix**) is a symmetric positive-definite $n \times n$ real matrix (usually it will be a diagonal matrix, but it need not be);
- K (the so-called **stiffness matrix**) is a symmetric $n \times n$ real matrix (usually it too will be positive-definite, but it need not be); and
- $\mathbf{0}$ denotes the zero vector.

We then try a solution of the form

$$\mathbf{x}(t) = \mathbf{e}\, e^{i\omega t} \tag{22}$$

where $\mathbf{e}$ is some fixed vector. This will solve (21) if (and only if)

$$(K - \omega^2 M)\mathbf{e} = \mathbf{0} \tag{23}$$

or equivalently

$$K\mathbf{e} = \omega^2 M\mathbf{e}\,. \tag{24}$$

This is a **generalized eigenvalue problem** (it would be the ordinary eigenvalue problem if M were the identity matrix). The **eigenvalues** — that is, the values of ω^2 for which there exists a solution $\mathbf{e} \neq \mathbf{0}$ — are the solutions of the nth-degree polynomial equation

$$\det(K - \omega^2 M) = 0\,, \tag{25}$$

and then the corresponding vectors $\mathbf{e}$ are the **eigenvectors**. The solution $\mathbf{x}(t) = \mathbf{e}\, e^{i\omega t}$ [or $\mathbf{x}(t) = \mathbf{e}\, \cos\omega t$] corresponding to such an **eigenpair** $(\omega, \mathbf{e})$ is a **normal mode**.

A generalized eigenvalue problem with a pair of real symmetric matrices, at least one of which is positive-definite, always has a basis of eigenvectors — that is, we can always find eigenvalues $\lambda_1, \ldots, \lambda_n$ and corresponding eigenvectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ such that

(a) $K\mathbf{e}_j = \lambda_j M\mathbf{e}_j$ for $j = 1, \ldots, n$; and

(b) $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is a basis of $\mathbb{R}^n$.

Let us prove this as follows:

Lemma 1 *Every symmetric positive-definite real matrix has a symmetric positive-definite square root. That is, if M is a symmetric positive-definite real $n \times n$ matrix, then there exists another symmetric positive-definite real $n \times n$ matrix, which we shall denote $M^{1/2}$, such that $M^{1/2}M^{1/2} = M$. (In fact this matrix $M^{1/2}$ is unique, but we shall not need this fact.)*

PROOF. Basic linear algebra tells us that any symmetric real matrix can be diagonalized by an orthogonal transformation: that is, there exists a matrix R satisfying $R^{\mathrm{T}}R = RR^{\mathrm{T}} = I$ (that is the definition of "orthogonal matrix") and $R^{\mathrm{T}}MR = D$, where D is a diagonal matrix. In fact, $D = \mathrm{diag}(m_1, \ldots, m_n)$, where $m_1, \ldots, m_n$ are the eigenvalues of M. In our case, the matrix M is positive-definite, so all the eigenvalues $m_1, \ldots, m_n$ are > 0; in particular, they have square roots. We can therefore define the matrix $D^{1/2} = \mathrm{diag}(m_1^{1/2}, \ldots, m_n^{1/2})$, which manifestly satisfies $D^{1/2}D^{1/2} = D$.

Now define $M^{1/2} = RD^{1/2}R^{\mathrm{T}}$. We have

$$\begin{aligned} M^{1/2}M^{1/2} &= RD^{1/2}R^{\mathrm{T}}RD^{1/2}R^{\mathrm{T}} \\ &= RD^{1/2}D^{1/2}R^{\mathrm{T}} \\ &= RDR^{\mathrm{T}} \\ &= R(R^{\mathrm{T}}MR)R^{\mathrm{T}} \\ &= M \end{aligned}$$

where we used $R^{\mathrm{T}}R = I$, then $D^{1/2}D^{1/2} = D$, then $D = R^{\mathrm{T}}MR$, and finally $RR^{\mathrm{T}} = I$. (You should check this carefully!)

I leave it to you to verify that $M^{1/2} = RD^{1/2}R^{\mathrm{T}}$ is symmetric.

Finally, since $M^{1/2} = RD^{1/2}R^{\mathrm{T}}$ with $D^{1/2}$ positive-definite and R nonsingular, it follows that $M^{1/2}$ is positive-definite as well. (You should go back to the definition of "positive-definite matrix" and verify this assertion too.) $\square$

Let us now show that a pair of real quadratic forms, one of which is positive-definite, can be simultaneously diagonalized:

Proposition 2 *Let M and K be a real symmetric $n \times n$ matrices, with M positive-definite. Then there exists a nonsingular real $n \times n$ matrix N such that $N^{\mathrm{T}}MN = I$ (the identity matrix) and $N^{\mathrm{T}}KN = \Lambda$, where Λ is a diagonal matrix.*

PROOF. Let $M^{1/2}$ be the symmetric positive-definite square root of M whose existence is guaranteed by the Lemma. Since $M^{1/2}$ is positive-definite, it is invertible. Then $L = (M^{1/2})^{-1}K(M^{1/2})^{-1}$ is a real symmetric matrix, so it can be diagonalized by an orthogonal transformation: that is, there exists a matrix R satisfying $R^{\mathrm{T}}R = RR^{\mathrm{T}} = I$ and $R^{\mathrm{T}}LR = \Lambda$, where Λ is a diagonal matrix. Now define $N = (M^{1/2})^{-1}R$. We have $N^{\mathrm{T}} = R^{\mathrm{T}}(M^{1/2})^{-1}$ (why?). Then $N^{\mathrm{T}}KN = \Lambda$ by construction (why?), and

$$\begin{aligned}
N^{\mathrm{T}}MN &= R^{\mathrm{T}}(M^{1/2})^{-1}M(M^{1/2})^{-1}R \\
&= R^{\mathrm{T}}(M^{1/2})^{-1}M^{1/2}M^{1/2}(M^{1/2})^{-1}R \\
&= R^{\mathrm{T}}R \\
&= I\ .
\end{aligned}$$

□

Corollary 3 *Let M and K be a real symmetric $n \times n$ matrices, with M positive-definite. Then there exist real numbers $\lambda_1, \ldots, \lambda_n$ and vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n \in \mathbb{R}^n$ such that*

(a) $K\mathbf{e}_j = \lambda_j M\mathbf{e}_j$ for $j = 1, \ldots, n$; and

(b) $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is a basis of $\mathbb{R}^n$.

PROOF. Let N be the matrix whose existence is guaranteed by the Proposition, and let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be its columns. Since N is nonsingular, its columns are linearly independent, hence form a basis of $\mathbb{R}^n$. Obviously N^{T} is also nonsingular, hence invertible, and the Proposition tells us that

$$\begin{aligned}
MN &= (N^{\mathrm{T}})^{-1} \\
KN &= (N^{\mathrm{T}})^{-1}\Lambda
\end{aligned}$$

(why?). Here Λ is a diagonal matrix; let $\lambda_1, \ldots, \lambda_n$ be its diagonal entries. Now, the jth column of MN is $M\mathbf{e}_j$ (why?), so the jth column of $(N^{\mathrm{T}})^{-1}$ is $M\mathbf{e}_j$. It follows that the jth column of $(N^{\mathrm{T}})^{-1}\Lambda$ is $\lambda_j M\mathbf{e}_j$ (why?). Since the jth column of KN is $K\mathbf{e}_j$, this proves that $K\mathbf{e}_j = \lambda_j M\mathbf{e}_j$. □

This manipulation of matrices is quick but perhaps a bit abstract. Here is a more direct proof of the Corollary that analyzes directly the generalized eigenvalue problem:

ALTERNATE PROOF OF COROLLARY 3. Let $M^{1/2}$ be the symmetric positive-definite square root of M whose existence is guaranteed by the Lemma. We can then rewrite

$$K\mathbf{e} = \lambda M\mathbf{e} \tag{26}$$

as

$$K\mathbf{e} = \lambda M^{1/2}M^{1/2}\mathbf{e}\ . \tag{27}$$

Defining $\mathbf{f} = M^{1/2}\mathbf{e}$, we have $\mathbf{e} = (M^{1/2})^{-1}\mathbf{f}$ [note that $M^{1/2}$ is invertible because it is positive-definite] and hence the equation can be rewritten as

$$K(M^{1/2})^{-1}\mathbf{f} \;=\; \lambda M^{1/2}\mathbf{f}\;. \tag{28}$$

And we can left-multiply both sides by $(M^{1/2})^{-1}$ to obtain

$$(M^{1/2})^{-1}K(M^{1/2})^{-1}\mathbf{f} \;=\; \lambda\mathbf{f} \tag{29}$$

[note that this operation is reversible because $(M^{1/2})^{-1}$ is invertible]. So we now have an *ordinary* eigenvalue problem for the symmetric real matrix $(M^{1/2})^{-1}K(M^{1/2})^{-1}$. This matrix has eigenvalues $\lambda_1, \ldots, \lambda_n$ and a corresponding basis of linearly independent eigenvectors $\mathbf{f}_1, \ldots, \mathbf{f}_n$. Defining $\mathbf{e}_j = (M^{1/2})^{-1}\mathbf{f}_j$, a simple calculation shows that

$$K\mathbf{e}_j \;=\; \lambda_j M\mathbf{e}_j \qquad \text{for } j = 1, \ldots, n\;. \tag{30}$$

And $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is a basis of $\mathbb{R}^n$ because $\{\mathbf{f}_1, \ldots, \mathbf{f}_n\}$ is a basis of $\mathbb{R}^n$ and the matrix $(M^{1/2})^{-1}$ is nonsingular. $\square$

4 Another example: n masses with springs

Now let us generalize the example of Section 2 by considering a chain of n particles, each of mass m, joined by $n+1$ springs, each of spring constant k, between a pair of walls. Then the equations of motion are

$$\begin{aligned}
m\ddot{x}_1 &= -kx_1 + k(x_2 - x_1) && \text{(31a)}\\
m\ddot{x}_2 &= k(x_1 - x_2) + k(x_3 - x_2) && \text{(31b)}\\
m\ddot{x}_3 &= k(x_2 - x_3) + k(x_4 - x_3) && \text{(31c)}\\
&\;\vdots\\
m\ddot{x}_{n-1} &= k(x_{n-2} - x_{n-1}) + k(x_n - x_{n-1}) && \text{(31d)}\\
m\ddot{x}_n &= k(x_{n-1} - x_n) - kx_n && \text{(31e)}
\end{aligned}$$

(You should check this carefully and make sure you understand this, including all the signs!) This system of differential equations can be written compactly in the form

$$M\ddot{\mathbf{x}} + K\mathbf{x} \;=\; \mathbf{0} \tag{32}$$

where $M = mI$ and the matrix K has entries $2k$ on the diagonal and $-k$ just above and below the diagonal (and zero entries everywhere else): that is, $K = kL$ where

$$L \;=\; \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix} \tag{33}$$

is called the **one-dimensional discrete Laplace matrix** (with Dirichlet boundary conditions at the endpoints).[2] The eigenvalues $\lambda = \omega^2$ of our generalized eigenvalue problem are simply k/m times the eigenvalues of the matrix L.

How can we find the eigenvalues and eigenvectors of L? This is not so obvious; it requires a bit of cleverness. Let us start by observing that the equation $L\mathbf{f} = \mu\mathbf{f}$ for the eigenvector $\mathbf{f} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}$ can be written in the simple form

$$-f_{s-1} + 2f_s - f_{s+1} \;=\; \mu f_s \qquad \text{for } s = 1, \ldots, n \tag{34}$$

if we simply define $f_0 = 0$ and $f_{n+1} = 0$. (Why?) Now this is a linear constant-coefficient difference equation; and by analogy with linear constant-coefficient *differential* equations, we might expect the solutions of (34) to be linear combinations of (complex) exponentials, e.g.

$$f_s \;=\; e^{i\alpha s} \tag{35}$$

where we can always take $-\pi < \alpha \le \pi$ (why?). Plugging the guess (35) into (34), we see that this guess indeed solves (34) provided that α and μ are related by

$$2 - 2\cos\alpha \;=\; \mu\ . \tag{36}$$

(You should check this carefully!) In particular we must have $0 \le \mu \le 4$. Note that to each allowed value of μ there corresponds a *pair* of allowed values of α — namely, a value $\alpha > 0$ and its negative — because cos is an even function.[3] So any linear combination of the two solutions $e^{i\alpha s}$ and $e^{-i\alpha s}$ is also a solution for the given value of μ; in particular, any linear combination of $\sin(\alpha s)$ and $\cos(\alpha s)$ is a solution.

But we are not done yet: we have solved the difference equation (34), but we have not yet dealt with the "boundary conditions" $f_0 = 0$ and $f_{n+1} = 0$. The condition $f_0 = 0$ can be satisfied simply by choosing the solution

$$f_s \;=\; \sin(\alpha s) \tag{37}$$

(why?). And the condition $f_{n+1} = 0$ can then be satisfied by making sure that $(n+1)\alpha$ is a multiple of π, i.e.

$$\alpha \;=\; \frac{j\pi}{n+1} \qquad \text{for some integer } j\ . \tag{38}$$

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[2] L is called the discrete Laplace matrix because it is the discrete analogue of the Laplacian operator $-d^2/dx^2$ in one dimension or $-\dfrac{\partial^2}{\partial x_1^2} - \cdots - \dfrac{\partial^2}{\partial x_n^2}$ in n dimensions. Indeed, if you were to try to solve on the computer a differential equation involving the operator $-d^2/dx^2$, you would probably discretize space (i.e. replace continuous space by a mesh of closely spaced points) and replace the operator $-d^2/dx^2$ by the matrix L or something similar.

[3] There are two exceptions: $\mu = 0$ corresponds only to $\alpha = 0$, and $\mu = 4$ corresponds only to $\alpha = \pi$.

We can't take $j = 0$, because that would make $\mathbf{f}$ identically zero; but we can take any integer j from 1 up to n. We have thus obtained eigenvectors $\mathbf{f}_1, \ldots, \mathbf{f}_n$ for the matrix L, given by

$$(\mathbf{f}_j)_s \;=\; \sin\!\left(\frac{\pi j s}{n+1}\right) , \tag{39}$$

with corresponding eigenvalues

$$\mu_j \;=\; 2 - 2\cos\!\left(\frac{\pi j}{n+1}\right) \tag{40a}$$

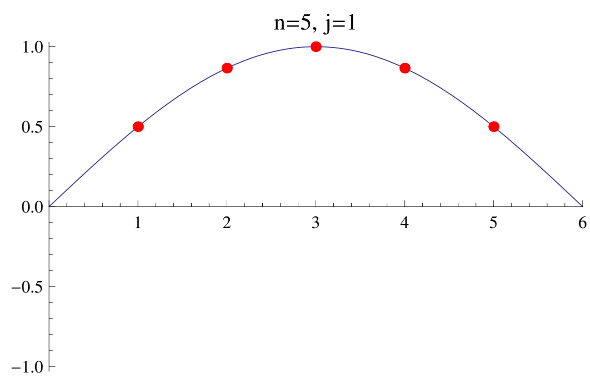
$$\;=\; 4\sin^2\!\left(\frac{\pi j}{2(n+1)}\right) . \tag{40b}$$

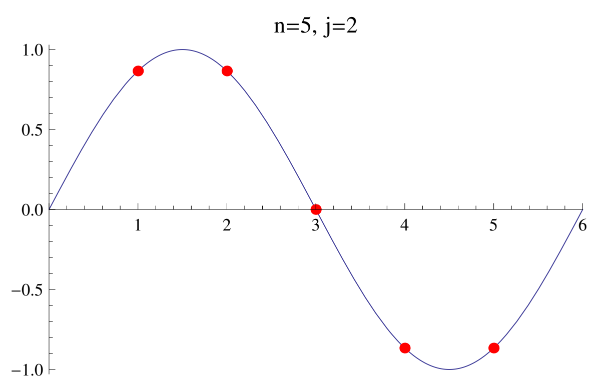
Since there are n of these and they are linearly independent (they must be because the values μ_j are all different!), we conclude that we have found a complete set of eigenvectors for the matrix L.[4]

Physically, these eigenvectors are **standing waves**. To see this, let us make some plots for $n = 5$ and $j = 1, 2, 3, 4, 5$. For each value of j, we first plot the function

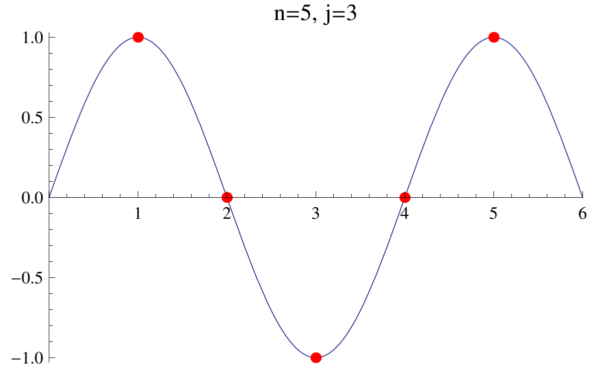
$$f_j(s) \;=\; \sin\!\left(\frac{\pi j s}{n+1}\right) \tag{41}$$

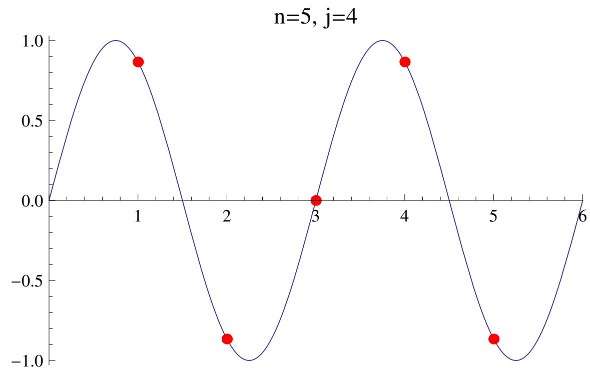
for *real* values of s in the interval $0 \le s \le n + 1$ (this shows most clearly the "standing wave"); then we indicate the points corresponding to $s = 1, 2, \ldots, n$, which are the entries in the eigenvector $\mathbf{f}_j$.

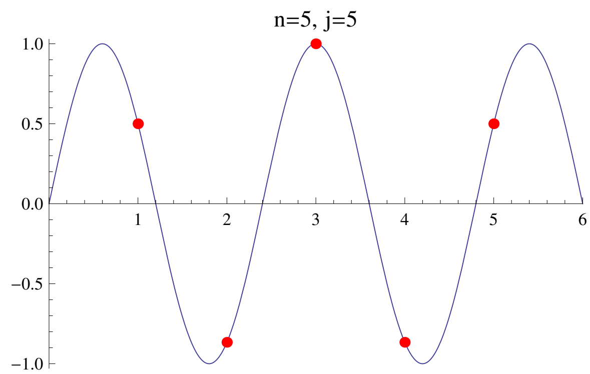




[4]It follows, in particular, that nothing new is obtained by going outside the range $1 \le j \le n$. For instance, $j = n+1$ again yields the zero function, $j = n+2$ yields a multiple of what $j = n$ yields, $j = n+3$ yields a multiple of what $j = n-1$ yields, and so forth.







Next week we will look at the limit $n \to \infty$ of this problem — namely, waves on a string of length L — and we will find standing-wave solutions corresponding to the functions

$$f_j(s) \;=\; \sin\!\left(\frac{\pi j s}{L}\right) \tag{42}$$

where s is now a *real* number satisfying $0 \le s \le L$, and j is now an *arbitrarily large* positive integer.