MATHEMATICS 1302 (Applied Mathematics 2)
YEAR 2008–2009, TERM 2

HANDOUT #7: MOTION CONSTRAINED TO A MOVING CURVE

1 Motion constrained to a moving curve: General case

Last week we discussed motion constrained to a fixed planar curve. We saw that the easiest way to derive the equation of motion was to write the conservation-of-energy equation; but we also showed how to obtain the equation of motion directly from $\mathbf{F} = m\mathbf{a}$.

Now we wish to consider motion constrained to a *moving* curve (e.g. a bead sliding along a moving wire). In this case we *cannot* use conservation of energy, because the normal force associated to a *moving* constraint *can* do work (as you saw explicitly in Problem 5 of Problem Set #3). We shall therefore go back to $\mathbf{F} = m\mathbf{a}$.

So consider a wire whose position at time t is given by a specified vector-valued function $\mathbf{f}(\lambda, t)$, where λ is a parameter indicating location along the curve (it could be the arc length s, but need not be). And consider a bead sliding along the wire, with position $\lambda(t)$ at time t. Then the position of the bead in space at time t is

$$\mathbf{r}(t) \;=\; \mathbf{f}(\lambda(t), t) \;. \tag{1}$$

The bead's velocity is therefore (using the chain rule)

$$\mathbf{v}(t) \;=\; \frac{d\mathbf{r}(t)}{dt} \;=\; \frac{\partial \mathbf{f}}{\partial \lambda}(\lambda(t), t)\, \dot{\lambda} \;+\; \frac{\partial \mathbf{f}}{\partial t}(\lambda(t), t) \;, \tag{2}$$

where I have stressed that f and all its derivatives are to be evaluated at $(\lambda(t), t)$. By the same method, the bead's acceleration is

$$\begin{aligned} \mathbf{a}(t) \;=\; \frac{d\mathbf{v}(t)}{dt} \;&=\; \frac{\partial^2 \mathbf{f}}{\partial \lambda}\, \dot{\lambda}^2 \;+\; \frac{\partial^2 \mathbf{f}}{\partial \lambda \partial t}\, \dot{\lambda} \;+\; \frac{\partial \mathbf{f}}{\partial \lambda}\, \ddot{\lambda} \;+\; \frac{\partial^2 \mathbf{f}}{\partial t \partial \lambda}\, \dot{\lambda} \;+\; \frac{\partial^2 \mathbf{f}}{\partial t^2} \\ &=\; \frac{\partial \mathbf{f}}{\partial \lambda}\, \ddot{\lambda} \;+\; \frac{\partial^2 \mathbf{f}}{\partial \lambda}\, \dot{\lambda}^2 \;+\; 2\frac{\partial^2 \mathbf{f}}{\partial \lambda \partial t}\, \dot{\lambda} \;+\; \frac{\partial^2 \mathbf{f}}{\partial t^2} \end{aligned} \tag{3}$$

(please make sure you understand the logic of these calculations!).

Now suppose that the bead slides *frictionlessly* along the wire, subject to an external force $\mathbf{F}_{\rm ext}$ (e.g. the earth's gravity). We can then write

$$m\mathbf{a} \;=\; \mathbf{N} + \mathbf{F}_{\rm ext} \tag{4}$$

where $\mathbf{N}$ is the normal force exerted by the wire on the bead. The key fact about $\mathbf{N}$ is that it is directed *perpendicular* to the wire (that is what we *mean* by "slides frictionlessly"). So let us introduce the tangent vector to the wire,

$$\mathbf{e}_\parallel \;=\; \frac{\partial \mathbf{f}}{\partial \lambda} \tag{5}$$

(please note that this is *not* in general a unit vector, unless λ is arc length, but no matter). Then we have $\mathbf{N} \cdot \mathbf{e}_{\|} = 0$, and hence the equation of motion can be written as

$$m\mathbf{a} \cdot \mathbf{e}_{\|} \;=\; \mathbf{F}_{\rm ext} \cdot \mathbf{e}_{\|} \;. \tag{6}$$

By using the specific form of the function $\mathbf{f}(\lambda, t)$ for a given problem, we can obtain a second-order ordinary differential equation for the unknown function $\lambda(t)$.

Rather than pursuing this general case any further, we shall concentrate on the specific case of a rotating wire.

2 Cylindrical coordinates

Cylindrical coordinates (r, θ, z) in three-dimensional space are a trivial generalization of plane polar coordinates (r, θ). We simply use plane polar coordinates (r, θ) to rewrite the coordinates (x, y) in the usual way, namely

$$x \;=\; r\cos\theta \tag{7a}$$
$$y \;=\; r\sin\theta \tag{7b}$$

and we leave z as is. The formulae for velocity and acceleration in cylindrical coordinates are likewise an obvious generalization of the formulae in plane polar coordinates:

$$\mathbf{v} \;=\; \dot{r}\widehat{e}_r \,+\, r\dot{\theta}\widehat{e}_\theta \,+\, \dot{z}\widehat{\mathbf{e}}_z \tag{8a}$$
$$\mathbf{a} \;=\; (\ddot{r} - r\dot{\theta}^2)\widehat{e}_r \,+\, (r\ddot{\theta} + 2\dot{r}\dot{\theta})\widehat{e}_\theta \,+\, \ddot{z}\widehat{\mathbf{e}}_z \tag{8b}$$

3 Motion constrained to a rotating wire

Consider a wire lying in the x–z plane, given by the formula

$$x \;=\; g(\lambda) \tag{9a}$$
$$z \;=\; h(\lambda) \tag{9b}$$

where g and h are specified functions, and λ is a parameter indicating location along the wire (it could be the arc length s, but need not be). Now rotate this wire around the z axis at angular frequency ω; its position $\mathbf{f}(\lambda, t)$ is then given by

$$x \;=\; g(\lambda)\cos\omega t \tag{10a}$$
$$y \;=\; g(\lambda)\sin\omega t \tag{10b}$$
$$z \;=\; h(\lambda) \tag{10c}$$

Because of the rotational symmetry of this problem, it is natural to use cylindrical coordinates to analyze it. In cylindrical coordinates, the wire's position is given by

$$r \;=\; g(\lambda) \tag{11a}$$
$$\theta \;=\; \omega t \tag{11b}$$
$$z \;=\; h(\lambda) \tag{11c}$$

Now consider a bead sliding along this rotating wire, with position $\lambda(t)$ at time t. We have

$$\dot{r} = g'(\lambda)\,\dot{\lambda} \tag{12a}$$
$$\ddot{r} = g''(\lambda)\,\dot{\lambda}^2 + g'(\lambda)\,\ddot{\lambda} \tag{12b}$$
$$\dot{\theta} = \omega \tag{12c}$$
$$\ddot{\theta} = 0 \tag{12d}$$
$$\dot{z} = h'(\lambda)\,\dot{\lambda} \tag{12e}$$
$$\ddot{z} = h''(\lambda)\,\dot{\lambda}^2 + h'(\lambda)\,\ddot{\lambda} \tag{12f}$$

so that the bead's acceleration is given by

$$\mathbf{a} = [g'(\lambda)\ddot{\lambda} + g''(\lambda)\dot{\lambda}^2 - \omega^2 g(\lambda)]\widehat{e}_r + 2\omega g'(\lambda)\,\dot{\lambda}\widehat{e}_\theta + [h'(\lambda)\ddot{\lambda} + h''(\lambda)\dot{\lambda}^2]\widehat{\mathbf{e}}_z\,. \tag{13}$$

Now, a unit vector tangent to the wire is

$$\widehat{\mathbf{e}}_\parallel = \frac{g'(r)\widehat{e}_r + h'(r)\widehat{\mathbf{e}}_z}{\sqrt{g'(r)^2 + h'(r)^2}}\,, \tag{14}$$

while unit vectors perpendicular to to the wire are given by

$$\widehat{\mathbf{e}}_\perp = \frac{-h'(r)\widehat{e}_r + g'(r)\widehat{\mathbf{e}}_z}{\sqrt{g'(r)^2 + h'(r)^2}} \tag{15}$$

and $\widehat{e}_\theta$. So we can write the normal force exerted by the wire on the bead as

$$\mathbf{N} = N_\perp \widehat{\mathbf{e}}_\perp + N_\theta \widehat{e}_\theta\,. \tag{16}$$

With external force $\mathbf{F}_{\rm ext}$, Newton's Second Law gives

$$m\mathbf{a} = \mathbf{N} + \mathbf{F}_{\rm ext}\,. \tag{17}$$

To get the equation of motion of the bead, we take the dot product of (17) with $\widehat{\mathbf{e}}_\parallel$; then $\mathbf{N}$ drops out and we get

$$m\mathbf{a}\cdot\widehat{\mathbf{e}}_\parallel = \mathbf{F}_{\rm ext}\cdot\widehat{\mathbf{e}}_\parallel\,, \tag{18}$$

or explicitly

$$m[g'(\lambda)\ddot{\lambda} + g''(\lambda)\dot{\lambda}^2 - \omega^2 g(\lambda)]g'(r) + m[h'(\lambda)\ddot{\lambda} + h''(\lambda)\dot{\lambda}^2]h'(r) = \mathbf{F}_{\rm ext}\cdot[g'(r)\widehat{e}_r + h'(r)\widehat{\mathbf{e}}_z]\,. \tag{19}$$

This looks ugly, but it usually simplifies in a specific problem when one inserts the specific functions $g(\lambda)$ and $h(\lambda)$.

Example of $\mathbf{F}_{\rm ext}$: If the bead is sliding under gravity, we have $\mathbf{F}_{\rm ext} = -mg\widehat{\mathbf{e}}_z$.

We can then, if we wish, extract the normal forces $N_\perp$ and N_θ by taking the dot product of (17) with $\widehat{\mathbf{e}}_\perp$ and $\widehat{e}_\theta$, respectively.

I will leave you to apply this general theory in Problem 1 of Problem Set #7 (= Problem 5 of Problem Set #6).