MATHEMATICS 1302 (Applied Mathematics 2)
YEAR 2008–2009, TERM 2

HANDOUT #6: MORE ON CENTRAL-FORCE MOTION

This handout is intended to complement the discussion of central-force motion contained in Kleppner and Kolenkow (K+K), Chapter 9. (Don't forget to read also Section 1.9 and Example 6.3 in K+K.)

1 Reduction to a one-dimensional problem (review)

The Newtonian equations of motion in polar coordinates for a particle of mass m subject to a central force $\mathbf{F}(r) = F(r)\,\widehat{e}_r$ are

$$m(\ddot{r} - r\dot{\varphi}^2) \;=\; F(r) \qquad \text{(1a)}$$

$$m(r\ddot{\varphi} + 2\dot{r}\dot{\varphi}) \;=\; 0 \qquad \text{(1b)}$$

As discussed in K+K, Section 9.3, these equations can be integrated by observing that the angular momentum

$$L \;=\; m\mathbf{r}\times\mathbf{v} \;=\; mr^2\dot{\varphi} \qquad (2)$$

and the total energy

$$E \;=\; \tfrac{1}{2}m\mathbf{v}^2 + U(r) \;=\; \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2) + U(r) \qquad (3)$$

are constants of motion (i.e. $dL/dt = 0$ and $dE/dt = 0$). It follows that

$$\tfrac{1}{2}m\dot{r}^2 \;+\; \frac{L^2}{2mr^2} \;+\; U(r) \;=\; E\;, \qquad (4)$$

which is mathematically identical to the energy-conservation equation for a particle moving in one dimension subject to the "effective potential energy"

$$U_{\rm eff}(r) \;=\; U(r) \;+\; \frac{L^2}{2mr^2}\;. \qquad (5)$$

In particular, eq. (4) can be written as

$$\frac{dr}{dt} \;=\; \pm\sqrt{\frac{2}{m}[E - U_{\rm eff}(r)]} \qquad (6)$$

and hence solved by separation of variables:

$$t \;=\; t_0 \pm \int\limits_{r_0}^{r} \frac{dr'}{\sqrt{\dfrac{2}{m}[E - U_{\rm eff}(r')]}}\;. \qquad (7)$$

From this one can in principle determine the motion $r(t)$. And plugging this back into the angular-momentum equation

$$\frac{d\varphi}{dt} = \frac{L}{mr^2} , \tag{8}$$

one can in principle find $\varphi(t)$ as well:

$$\varphi(t) = \varphi_0 + \int_{t_0}^{t} \frac{L}{m\, r(t')^2}\, dt' . \tag{9}$$

In practice it is often impossible to perform explicitly the integrals and inversions here (at least in terms of elementary functions); in this case we have to resort to numerical solutions.

The qualitative analysis of central-force motions is discussed in K+K, Section 9.5.

2 Equation of the orbit: First-order version

There are some cases in which it is hopeless to find $r(t)$ and $\varphi(t)$ explicitly in terms of elementary functions, but it is nevertheless possible to find the *orbit* $r(\varphi)$ explicitly. Indeed, the inverse-square force is an example of this, as we shall see.

K+K already observe (top p. 382) that dividing eq. (8) by eq. (6) yields

$$\frac{d\varphi}{dr} = \pm \frac{L}{mr^2 \sqrt{\dfrac{2}{m}[E - U_{\rm eff}(r)]}} , \tag{10}$$

from which one can in principle find the orbit $\varphi(r)$ by integration. This integral will be of the form

$$\int \frac{dr}{r^2 \sqrt{\rm stuff}} , \tag{11}$$

and the factor $dr/r^2 = -d(1/r)$ appearing here suggests that it might be useful to make the change of variables $u = 1/r$. Doing this, we have

$$\frac{d\varphi}{du} = \pm \frac{L}{m \sqrt{\dfrac{2}{m}[E - U_{\rm eff}(1/u)]}} \tag{12a}$$

$$= \pm \frac{L}{m \sqrt{\dfrac{2}{m}[E - U(1/u) - \dfrac{L^2}{2m} u^2]}} . \tag{12b}$$

This equation can be separated and integrated to give

$$\varphi = \varphi_0 \pm \int \frac{(L/m)\, du}{\sqrt{\dfrac{2}{m}[E - U(1/u) - \dfrac{L^2}{2m} u^2]}} . \tag{13}$$

In principle this gives $\varphi(u)$ and hence $\varphi(r)$. Note that the constants of motion L and E appear explicitly.

Remarks. 1. The physical meaning of the variable $u = 1/r$ can be clarified by throwing in an additional factor L/m, because we then have

$$\frac{L}{mr} \;=\; r\dot{\varphi} \;=\; v_\perp \;, \tag{14}$$

i.e. $v_\perp$ is the angular component of the particle's velocity $\mathbf{v} = \dot{r}\widehat{e}_r + v_\perp \widehat{e}_\varphi$. So u is proportional to $v_\perp$.

2. The $\pm$ sign in eq. (12) depends on whether we are considering the outgoing or incoming part of the orbit. It can be determined from

$$\frac{du}{d\varphi} \;=\; \frac{du}{dr}\frac{dr}{d\varphi} \;=\; -\frac{1}{r^2}\frac{dr}{d\varphi} \;=\; -\frac{m}{L}\frac{d\varphi}{dt}\frac{dr}{d\varphi} \;=\; -\frac{m}{L}\frac{dr}{dt} \;. \tag{15}$$

3 Equation of the orbit: Second-order version

Here is an alternate (and sometimes useful) approach to the orbit equation, which yields a *second-order* differential equation for the function $u(\varphi)$. Let us start by turning eq. (12) upside-down to yield

$$\frac{du}{d\varphi} \;=\; \pm\sqrt{\frac{2m}{L^2}[E - U(1/u) - \frac{L^2}{2m}u^2]} \tag{16}$$

and then squaring it to yield

$$\left(\frac{du}{d\varphi}\right)^2 + u^2 + \frac{2m}{L^2}U(1/u) \;=\; \frac{2mE}{L^2} \;. \tag{17}$$

Now differentiate both sides with respect to φ: we get

$$2\frac{d^2u}{d\varphi^2}\frac{du}{d\varphi} + 2u\frac{du}{d\varphi} - \frac{2m}{L^2u^2}U'(1/u)\frac{du}{d\varphi} \;=\; 0 \tag{18}$$

where U' denotes the derivative of U with respect to its argument (namely, dU/dr). We can now pull out a common factor $2\,du/d\varphi$ to obtain

$$\frac{d^2u}{d\varphi^2} + u \;=\; \frac{m}{L^2u^2}U'(1/u) \;=\; -\frac{m}{L^2u^2}F(1/u) \tag{19}$$

where we have used $F(r) = -U'(r)$. (Note that E disappeared in the differentiation, as it should, since it is a constant of integration obtained in the passage from a second-order differential equation to a first-order one. We have just reversed this process, so E disappears.) In summary, we have

$$\boxed{\frac{d^2u}{d\varphi^2} + u \;=\; -\frac{m}{L^2u^2}F(1/u)} \tag{20}$$

The orbit equation (20) is a second-order differential equation that is *mathematically* of the same form as the Newtonian equation of motion for a particle moving in one dimension, but with φ now replacing t as the independent variable and u replacing x as the dependent variable. In particular, the left-hand side looks like the equation of the harmonic oscillator, while the right-hand side provides an "anharmonic" term. There are two special cases in which this latter term is very simple:

- If $F = -k/r^2$ (inverse-square force), the right-hand side of eq. (20) is the *constant* km/L^2.
- If $F = -k/r^3$ (inverse-cube force), the right-hand side of eq. (20) is $(km/L^2)u$, i.e. *linear* in u.

In both these cases (and more generally when the force is a sum of inverse-square and inverse-cube terms), the orbit equation (20) is a *linear* constant-coefficient second-order differential equation and hence easily solvable.

4 An example

The second-order form of the orbit equation is particularly useful for finding the force law when the orbit is known, since we can rewrite eq. (20) as

$$F(1/u) \;=\; -\frac{L^2u^2}{m}\left(\frac{d^2u}{d\varphi^2} + u\right) . \tag{21}$$

Here is an amusing example:

Suppose that the orbit is a circle of radius a that *passes through* the origin (this is admittedly a very strange orbit!). If we place the center of the orbit at $(x, y) = (a, 0)$, then a bit of geometry (do it!) shows that the equation of the orbit in plane polar coordinates is

$$r \;=\; 2a\cos\varphi . \tag{22}$$

Let us now ask: What is the central force $F(r)$ that permits a particle to move with angular momentum L along this circle?

Inserting $u = (2a)^{-1}\sec\varphi$ into (21), we find after some algebra (you should check this!!)

$$F(1/u) \;=\; -\frac{L^2}{4ma^3}\sec^5\varphi \;=\; -\frac{8L^2a^2}{m}u^5 \tag{23}$$

or equivalently

$$F(r) \;=\; -\frac{8L^2a^2}{m}\frac{1}{r^5} . \tag{24}$$

So an attractive inverse-fifth-power force law $F(r) = -k/r^5$ $(k > 0)$ permits a circular orbit that passes through the center of force. Of course, the initial conditions have to be chosen just right for the particle to adopt this particular orbit.

5 Orbits for inverse-square forces

If the force law is inverse-square, $F(r) = -k/r^2$, then the second-order orbit equation (20) becomes

$$\frac{d^2u}{d\varphi^2} + u = \frac{km}{L^2} , \tag{25}$$

and its general solution is

$$u = \frac{km}{L^2} + A\cos(\varphi - \varphi_0) \tag{26}$$

where A and φ_0 are constants. Going back to the variable r, this yields

$$r = \frac{r_0}{1 + \epsilon\cos(\varphi - \varphi_0)} \tag{27}$$

where we have written

$$r_0 = \frac{L^2}{km} \tag{28}$$

$$\epsilon = \frac{AL}{k} \tag{29}$$

As discussed in K+K, Section 9.6 and Note 9.1, this is the equation in polar coordinates of a **conic section**; here r_0 is called the **semilatus rectum** and ϵ is called the **eccentricity**. In particular we have

$$\begin{aligned} \epsilon = 0&: \quad \text{circle} \\ 0 < \epsilon < 1&: \quad \text{ellipse} \\ \epsilon = 1&: \quad \text{parabola} \\ \epsilon > 1&: \quad \text{hyperbola} \end{aligned}$$

This result can alternatively be derived (as in K+K, Section 9.6) by using the first-order orbit equation (13). This approach has the advantage of relating the eccentricity to the total energy E:

$$\epsilon = \sqrt{1 + \frac{2EL^2}{mk^2}} . \tag{30}$$

It follows that

$$\begin{aligned} E < 0&: \text{ circle or ellipse} \\ E = 0&: \text{ parabola} \\ E > 0&: \text{ hyperbola} \end{aligned}$$

In interpreting these relations we must remember that we have chosen the zero of potential energy to lie at $r = \infty$, i.e. $U(r) = -k/r$. If the force is attractive ($k > 0$), this means that *all potential energies are negative*, and that locations closer to the origin have *more negative* potential energies. Whether the total energy is positive, negative or zero then depends on whether the kinetic energy (which is always nonnegative) is larger or smaller in magnitude than the potential energy (which is negative). More precisely, a positive total energy means that the particle can escape to infinity with a nonzero limiting speed; a zero total energy means that the particle can just barely escape to infinity with a zero limiting speed; and a negative total energy means that the particle's orbit is bounded (i.e. reaches a maximum distance from the origin $r_{\max} < \infty$).

You should study carefully K+K, Section 9.6 and Note 9.1 concerning the mathematics of conic sections and their application to inverse-square orbits.

6 Nearly circular orbits (for a general central force)

Let us now return to the case of a general central force $F(r)$. Consider the angular momentum L to be fixed once and for all. Recall that the second-order orbit equation (20) reads

$$\frac{d^2u}{d\varphi^2} + u \;=\; -\frac{m}{L^2u^2}F(1/u) \tag{31}$$

where $u = 1/r$. The simplest solution of this equation is a circular motion $u = \text{constant} = u_0$, where u_0 solves the algebraic equation

$$u_0 \;=\; -\frac{m}{L^2u_0^2}F(1/u_0) \tag{32}$$

for the given value of L. (Note that this implies that the force is at least locally attractive, i.e. $F(1/u_0) < 0$.)

Now suppose that, with the particle in this circular orbit, we give it a small radial impulse (which leaves L unchanged). Will u now oscillate around u_0? And if so, what will the orbit be?

To investigate small oscillations, we write

$$u(\varphi) \;=\; u_0 + \alpha(\varphi) \tag{33}$$

where α is considered small. We then expand the right-hand side of eq. (31) in Taylor series around $u = u_0$:

$$\begin{aligned} -\frac{m}{L^2u^2}F(1/u) \;&=\; -\frac{m}{L^2u_0^2}F(1/u_0) - \frac{m}{L^2}\left(\frac{d}{du}\frac{F(1/u)}{u^2}\right)_{u=u_0}(u-u_0) + O\Big((u-u_0)^2\Big) \\ &=\; u_0 - \frac{m}{L^2}\left(\frac{d}{du}\frac{F(1/u)}{u^2}\right)_{u=u_0}(u-u_0) + O\Big((u-u_0)^2\Big) \end{aligned} \tag{34}$$

where $O\Big((u-u_0)^2\Big)$ denotes terms of order $(u-u_0)^2$ and higher. Dropping all such terms, we obtain the differential equation

$$\frac{d^2\alpha}{d\varphi^2} + \omega^2\alpha \;=\; 0\,, \tag{35}$$

where we have defined

$$\omega^2 \equiv 1 + \frac{m}{L^2} \left(\frac{d}{du} \frac{F(1/u)}{u^2} \right)_{u=u_0} \tag{36a}$$

$$= 1 - \frac{2m}{L^2} \frac{F(1/u_0)}{u_0^3} - \frac{m}{L^2} \frac{F'(1/u_0)}{u_0^4} \tag{36b}$$

$$= 1 + 2 + \frac{F'(1/u_0)}{u_0 \, F(1/u_0)} \tag{36c}$$

$$= 3 + \frac{r_0 \, F'(r_0)}{F(r_0)} \; . \tag{36d}$$

[Here we have used eq. (32) in going from the second to the third line (check this!), and we have written $r_0 = 1/u_0$.] Eq. (35) is the differential equation of a simple harmonic oscillator with "frequency" ω, but with the angle φ playing the role of "time". Thus:

- If $\omega^2 > 0$, then the solutions are oscillatory,

  $$\alpha(\varphi) = A \cos[\omega(\varphi - \varphi_0)] \; , \tag{37}$$

  and the period of oscillation (an angle) is given by $2\pi/\omega$.

- If $\omega^2 < 0$, then the solutions are growing and decaying exponentials,

  $$\alpha(\varphi) = A e^{\sqrt{-\omega^2}\,\varphi} + B e^{-\sqrt{-\omega^2}\,\varphi} \; , \tag{38}$$

  and the orbit is *unstable*: that is, unless the initial conditions are perfectly tuned so that the amplitude of the growing exponential is exactly zero, the deviation from the circular orbit will grow exponentially with time, and the small-α approximation will soon fail.

Examples. 1. Consider an attractive inverse-nth-power force $F(r) = -k/r^n$ with $k > 0$. Then

$$\frac{r_0 \, F'(r_0)}{F(r_0)} = n \quad \text{independent of } r_0 \tag{39}$$

and hence

$$\omega^2 = 3 - n \; . \tag{40}$$

So the angle turned through in one complete oscillation is

$$\Phi = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{3-n}} \; . \tag{41}$$

- For $n = 2$ (inverse-square force), we have $\Phi = 2\pi$, and the orbit is closed (as we already know: it is an ellipse).
- For $n = -1$ (two-dimensional simple harmonic oscillator), we have $\Phi = \pi$. (Why is this correct? What is the orbit in this case?)

- As $n \uparrow 3$, we have $\Phi \uparrow \infty$.
- For $n > 3$, the orbit is unstable.

2. Consider a small perturbation of an inverse-square force: $F(r) = -k/r^2 - \lambda F_1(r)$ with λ small [here $F_1(r)$ is some specified function]. Then Φ will be slightly different from 2π, and the perihelion (point of closest approach to the origin) will *precess*: more precisely, it will advance if $\Phi > 2\pi$ and be retarded if $\Phi < 2\pi$. You will compute this, to first order in the small parameter λ, in the Problem Set; and you will apply it to compute the precession of the perihelion predicted by Einstein's general relativity.