MATHEMATICS 1302 (Applied Mathematics 2)
YEAR 2008–2009, TERM 2

PROBLEM SET #8 (LAST PROBLEM SET!)

This problem set is due at the *beginning* of lecture on Thursday 26 March (either morning or afternoon).

Topics: Coupled oscillations and normal modes. Standing waves on a linear chain.

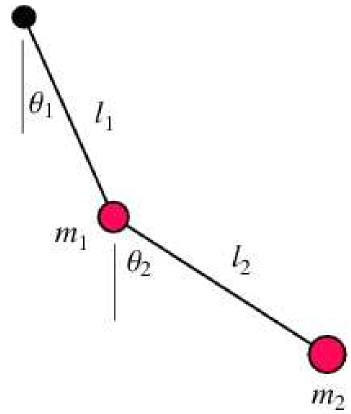
Readings:

- Handout #8: Coupled oscillations and normal modes.
- You might also wish to consult Dr. Baigent's notes on oscillations. I have made them available on the course website.

1. Consider a linear triatomic molecule (e.g. carbon dioxide), which we will model as a central particle of mass M connected on the left and right to particles of mass m, where each connection is via a spring of spring constant k and equilibrium length ℓ. Let us number these particles from left to right as 1,2,3, and let x_i be the displacement of particle $\#i$ from its equilibrium position. (We consider only one-dimensional motion along the given line.)
   (a) Find the equations of motion of this system.
   (b) Find the normal modes and, for each normal mode, describe the associated motion in words. What is the meaning of the normal mode with eigenfrequency 0?

2. Consider a chain of n particles (each of mass m) joined by n springs (each of spring constant k and equilibrium length ℓ) as follows: particle #1 is connected to a fixed left wall by spring #1, particle #2 is connected to particle #1 by spring #2, and so forth, until particle $\#n$ is connected to particle $\#n-1$ by spring $\#n$. So the situation is the same as that considered in Section 4 of Handout #8, *except that* the right wall and the rightmost spring are absent. Let x_i be the displacement of particle $\#i$ from its equilibrium position.
   (a) Find the equations of motion of this system.
   (b) For the case $n = 2$, find the normal modes.
   (c) *Optional:* Find the normal modes for general n. [*Hint:* Imitate the trick used in Section 4 of Handout #8, but with $f_{n+1} = f_n$ instead of $f_{n+1} = 0$.]
   (d) How would things change if this chain were hanging vertically in the earth's gravitational field, rather than horizontally?

3. A double pendulum consists of one pendulum attached to another, as shown:



The two rods are massless and rigid.

(a) Find the Cartesian coordinates (x_1, y_1) and (x_2, y_2) of the two masses in terms of θ_1 and θ_2. Take the the origin to be at the top support point.

(b) Compute $\ddot{x}_1$, $\ddot{y}_1$, $\ddot{x}_2$ and $\ddot{y}_2$ in terms of θ_1, θ_2 and their time derivatives.

(c) Find the forces acting on the two masses and, using your result from part (b), write the Newtonian equations of motion in terms of θ_1 and θ_2. Your equations will involve two unknown tensions.

(d) *Optional:* Eliminate the tensions to find a pair of coupled nonlinear differential equations for θ_1 and θ_2. [*Hint:* Work first on the $\ddot{x}_2$ and $\ddot{y}_2$ equations and form a linear combination of them to eliminate T_2. Then work on the equations for $m_1\ddot{x}_1$ + $m_2\ddot{x}_2$ and $m_1\ddot{y}_1 + m_2\ddot{y}_2$ (these combinations already eliminate T_2 — why?) and form a linear combination of them to eliminate T_1.] Show that these equations are

$$\begin{aligned} (m_1+m_2)l_1\ddot{\theta}_1 + m_2 l_2\ddot{\theta}_2\cos(\theta_1-\theta_2) + m_2 l_2\dot{\theta}_2^2\sin(\theta_1-\theta_2) + (m_1+m_2)g\sin\theta_1 &= 0 \\ m_2 l_2\ddot{\theta}_2 + m_2 l_1\ddot{\theta}_1\cos(\theta_1-\theta_2) - m_2 l_1\dot{\theta}_1^2\sin(\theta_1-\theta_2) + m_2 g\sin\theta_2 &= 0 \end{aligned}$$

[I am giving you these equations so that you'll be able to do parts (e) and (f) even if you had trouble doing parts (b)–(d).] Do these equations make sense in the limit $m_2/m_1 \to 0$?

(e) Find the linearized equations when θ_1 and θ_2 are assumed small.

(f) For the case $l_1 = l_2 = l$ (but m_1 and m_2 arbitrary), find the normal modes.

[**Remark.** The double pendulum is especially interesting when one does *not* make the small-angle approximation, but instead studies the full nonlinear equations. Then the motion can be *chaotic*, i.e. exhibit sensitive dependence to initial conditions. Computer simulations of the double pendulum can be found at numerous places on the web. One striking video of a real double pendulum can be found at `http://www.youtube.com/watch?v=Whvl6CikDxA`. Note that, in this video, friction causes the amplitude to very gradually decrease, so that one eventually leaves the chaotic regime and enters the small-oscillations regime.]