MATHEMATICS 1302 (Applied Mathematics 2)
YEAR 2008–2009, TERM 2

PROBLEM SET #6

This problem set is due at the *beginning* of problem class on Wednesday 11 March.

Topics: Central-force motion (continued). Motion constrained to a planar curve.

Readings:

- Finish the readings from last week.
- Dr. Baigent's 1302 notes on "Motion on a planar curve" (handout).

1. Do Problem 6 of Problem Set #5. (If you already did this problem last week, fine; otherwise just do it now. It will be assessed as part of *this week's* problem set.)

2. A particle P of mass m_1 lying on a frictionless horizontal table is joined by a massless string of length ℓ, passing through a smooth small hole O in the table, to a particle Q of mass m_2. At all times the string stays taut, with OQ vertical. Let (r, θ) be the position of particle P, using plane polar coordinates with the origin at O, and let z be the length of OQ.

   (a) Write the Newtonian equations of motion for r, θ and z, and write the constraint equation.

   (b) Eliminate z in favor of r, thereby obtaining a pair of coupled equations for r and θ.

   (c) Prove that the angular momentum of particle P is conserved, and exploit angular-momentum conservation to obtain a differential equation for r alone.

   (d) Show that, for any a satisfying $0 < a < \ell$, uniform circular motion at radius a (i.e. $r = a =$ constant) is possible at a suitable angular velocity $\dot{\theta}$, and find that value of $\dot{\theta}$.

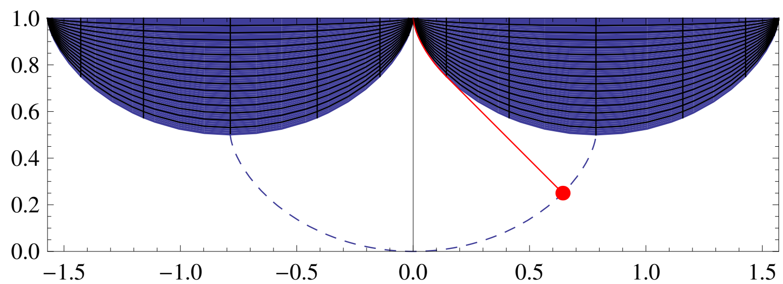
   (e) Find the frequency of small radial oscillations about the circular motion found in part (d).

3. Consider the following ingenious device: [Don't worry — it's not obvious *why* it's so ingenious until after you've solved the problem!]

A pendulum of length l is suspended from the cusp of a cycloid that is cut in a rigid support. The equation of this cycloid is

$$\begin{aligned} x &= \frac{l}{4}(\varphi - \sin\varphi) \\ z &= \frac{l}{4}(3 + \cos\varphi) \end{aligned}$$

where φ is a parameter. This is depicted below for the case $l = 1$:



(a) Prove that the path of the pendulum bob is also cycloidal and is given by

$$\begin{aligned} x &= \frac{l}{4}(\varphi + \sin\varphi) \\ z &= \frac{l}{4}(1 - \cos\varphi) \end{aligned}$$

Show that it is congruent to the first cycloid.

(b) Express the equation of motion of the pendulum in terms of the arc length s along the path of motion. Show that the oscillations are simple harmonic, and that the frequency of oscillation is $(g/l)^{1/2}$.

The wonderful thing about this device is that the oscillations are precisely *isochronous*, that is, the frequency of oscillation is independent of the amplitude of oscillation. Most oscillatory systems are *not* isochronous. Indeed, the ordinary simple pendulum is not isochronous, as we have seen in class. This is bad news if you are a seventeenth-century maker of grandfather clocks: you will somehow have to control the amplitude accurately, which is difficult. The isochronous pendulum neatly circumvents this problem.

All this was discovered by Christian Huygens in 1657, before Newton's *Principia*! Unfortunately, he never raked in the patent royalties he deserved; the invention never found much practical use.

Note also that your derivation can be done in reverse, to *discover* that the path of an isochronous pendulum must be a cycloid. It is amazing to realize that Huygens did this (presumably in this way) before calculus was even invented!

4. A cylinder of radius R is mounted on a table with its axis pointing upwards, and a point mass m subject to no external forces is attached to a massless unstretchable cord attached to the cylinder. Initially the cord is wound around the cylinder so that the mass is touching the cylinder. A tangentially-directed impulse now gives the mass an initial speed v_0, and the thing starts to unwind. Let φ be the angular location of the point where the cord loses contact with the cylinder, measured relative to the initial (fully wound) angular position of the mass.

   (a) Find the equation for the curve $\mathbf{r} = (x, y)$ along which the mass moves, parametrized by the angle φ.

   (b) Find the Newtonian equation of motion for the angle φ. [*Hint:* One easy way to do this is to find the position of the mass in terms of φ, and write the energy equation.]

   (c) Find φ as a function of time.

   (d) Find the tension in the cord as a function of time.

   (e) Find the angular momentum of the mass about the cylinder's axis, as a function of time. Is angular momentum conserved? Why or why not? Verify the torque–angular momentum theorem.

5. A smooth thin wire is bent into the shape of a parabola, $z = x^2/2a$, and is made to rotate with angular velocity ω about the z axis [i.e. about the point $x = 0$ on the wire]; here the $+z$ direction is of course oriented upwards. A bead of mass m then slides frictionlessly on the wire under the influence of gravity. Using cylindrical coordinates (ρ, φ, z), discuss the motion of the bead, as follows:

   (a) Write the ρ, φ and z components of Newton's equations of motion for the bead. Your equations will contain two unknown constraint forces.

   (b) Use the equations of constraint to eliminate all reference to φ, z, and their time derivatives as well as to the constraint forces. That is, you should obtain a differential equation for ρ alone.

   (c) Show that the total mechanical energy E of the bead is *not* conserved, and that the constraint force does work at a rate precisely dE/dt.

   (d) Show that the equation of motion found in part (b) can be integrated once by the usual trick of multiplying it by $\dot{\rho}$. What is the relation between this result and part (c)?

   (e) Integrate the equation of motion once more to get an "explicit" expression for t as a function of ρ (albeit in terms of an ugly integral).