MATHEMATICS 1302 (Applied Mathematics 2)
YEAR 2008–2009, TERM 2

PROBLEM SET #5

This problem set is due at the *beginning* of problem class on Wednesday 4 March.

Topics: Velocity and acceleration in polar coordinates. Central-force motion. Kepler's laws of planetary motion.

Readings:

- Kleppner and Kolenkow, Section 1.9 (handout).
- Kleppner and Kolenkow, Chapter 9 (handout). And don't forget Example 6.3, which is contained in a previous handout and which discusses the geometrical meaning of angular-momentum conservation.
- Handout #6: More on Central-Force Motion.
- You may also find it useful to read *last year's* 1302 notes on this subject, written by Dr. Baigent. I have made them available on the course website.

1. Kleppner and Kolenkow, Problem 9.8.

2. A particle of mass m moves under an attractive inverse-cube central force $F(r) = -\alpha/r^3$ with $\alpha > 0$. Initially the particle is at a distance a from the origin and is moving purely tangentially with speed v_0. Describe qualitatively the motion of the particle — note that there will be several different possible behaviors, depending on the values of a and v_0. [*Hint:* Find the angular momentum L, the total energy E, and the effective potential energy $U_{\text{eff}}(r)$. Then analyze qualitatively the corresponding one-dimensional motion problem.]

3. A spherical planet has radius R, and the speed of an object that moves in a circular orbit skimming its surface is v_c. Far away, a space probe is moving along a line that would carry it within a distance $3R$ of the planet's surface if the path were undeflected by gravity.
   (a) What initial speed v_0 must the probe have in order that its hyperbolic orbit just barely skim the planet's surface?
   (b) Write an equation for the orbit, letting $\varphi = 0$ be the point where the probe skims the planet's surface.
   (c) Sketch the orbit, and determine the angle through which the path of the probe is deflected.

4. (a) Sketch the curve represented in plane polar coordinates by $r = a\cos 2\varphi$.

   (b) Find the potential $V(r)$ that permits a particle to move with angular momentum L along the orbit $r = a\cos 2\varphi$.

5. A particle of mass m is subject to a repulsive force $F(r) = k/r^5$, where k is a positive constant. Initially it is at a very large distance from the force center and has a velocity whose magnitude is such that, if the particle were aimed directly at the center of force, the distance of closest approach to the force center would be a. Actually the particle is projected, with this same velocity, along a path that would cause it to pass within a distance b of the force center if it were not deflected. Find (in terms of k, a, b, m) the distance of closest approach to the force center, and the minimum speed the particle experiences.

6. Consider a particle of mass m moving in a central force $F(r) = -kr^{-2} - \lambda F_1(r)$. Compute, to first order in the small parameter λ, the precession of the perihelion (in radians per revolution) for a nearly circular orbit of radius R. Be sure to say whether the perihelion *advances* or is *retarded* at each revolution.

   Here are two important special cases:

   (a) Suppose that in addition to the Sun (mass M), the solar system contained dust of uniform density ρ. Compute the corresponding central force $F(r)$. [*Hint:* The calculation given in Kleppner + Kolenkow, Note 2.1 (see an old handout) shows that the gravitational force of a uniform spherical shell on a test mass located *outside* the shell is the same as if all the shell's mass were located at its center, while the force on a test mass located *inside* the shell is *zero*.] Show that the perihelion for a nearly circular orbit of radius R would be retarded, and compute the retardation per revolution.

   (b) According to general relativity, the orbit of a particle of mass m in the gravitational field of a star (or black hole) of mass M is identical to the Newtonian orbit in a central force
$$F(r) = -\frac{GMm}{r^2}\left[1 + \frac{3L^2}{m^2c^2r^2}\right]$$
   where L is the particle's angular momentum and c is the speed of light. (Note that L/mcr is dimensionless and equals $v_\perp/c$. So the general-relativistic correction is small whenever the particle's motion is nonrelativistic.) Show that the perihelion for a nearly circular orbit of radius R advances, and compute the advance per revolution.

   Now plug in $G = 6.673 \times 10^{-11}\,\mathrm{m}^3\mathrm{kg}^{-1}\mathrm{sec}^{-2}$, $M_{sun} = 1.99 \times 10^{30}\,\mathrm{kg}$, $c = 2.998 \times 10^8$ m/sec and

Planet	Semimajor Axis in Astronomical Units	Eccentricity
Mercury	0.387	0.206
Venus	0.723	0.007
Earth	1.000	0.017
Mars	1.524	0.093

where 1 AU = 1.496×10^{11} m. For Mercury, do you get the famous 43 seconds of arc per century? [Actually you'll be off by about 4%, because for non-circular orbits there is a correction factor $1 - \epsilon^2$ where ϵ is the eccentricity.]

You can find more information at `http://en.wikipedia.org/wiki/Tests_of_general_relativity#Perihelion_precession_of_Mercury` and `http://en.wikipedia.org/wiki/Kepler_problem_in_general_relativity`