MATHEMATICS 1302 (Applied Mathematics 2)
YEAR 2008–2009, TERM 2

PROBLEM SET #4

NO CLASSES the week of 16–20 February (Reading Week).

This problem set is due at the *beginning* of problem class on Wednesday 25 February (the week after Reading Week).

Topics: Mathematics of conservative and nonconservative forces; connection with gradient, curl and Stokes' theorem. Collision problems in one and two dimensions.

Readings:

- Kleppner and Kolenkow, Chapter 5 (handout).
- You may also wish to look at a textbook on vector calculus concerning the gradient, curl and Stokes' theorem. You will study these topics in more detail in MATH 1402 (see the notes for Weeks 5–7 at `http://www.ucl.ac.uk/~ucahdrb/MATH1402/index.htm`).
- Kleppner and Kolenkow, Section 4.14 (handout from last week).

1. A satellite of mass m is in a circular orbit of radius r around the earth (mass M_{E}). Compute
   (a) the satellite's speed v;
   (b) the satellite's kinetic energy K;
   (c) the satellite's potential energy U [assuming the usual convention that the zero of potential energy is placed at $r = \infty$];
   (d) the satellite's total energy E;
   (e) the satellite's angular momentum L.

   Now assume that friction with the earth's atmosphere causes the satellite to slowly lose energy, while maintaining a circular orbit (the radius of which may be slowly changing). Say whether each of the following will increase, decrease or stay the same during this process:

   (i) the satellite's total energy E;
   (ii) the satellite's kinetic energy K;

(iii) the satellite's potential energy U;

(iv) the satellite's speed v;

(v) the satellite's orbital radius r;

(vi) the satellite's angular momentum L.

[*Hint:* First answer question (i), being very careful about positive and negative numbers. Then use your results from (a)–(e) to express all the other quantities in terms of E. Does your answer agree with what you know about what happens to satellites as they lose energy? Does your answer agree with the work-energy theorem and the torque-angular momentum theorem?]

2. Kleppner and Kolenkow, Problem 4.14.

3. For each of the following force laws, find a potential energy function if one exists, or else prove that none exists. (If the force contains parameters such as $A, B, \ldots$, then consider all possible values of those parameters.)

   (a) $\mathbf{F}(x, y) = Ax^2 \widehat{\mathbf{e}}_x + By \widehat{\mathbf{e}}_y$

   (b) $\mathbf{F}(x, y) = Axy \widehat{\mathbf{e}}_x + By \widehat{\mathbf{e}}_y$

   (c) $\mathbf{F}(x, y) = -A \dfrac{x \widehat{\mathbf{e}}_x + y \widehat{\mathbf{e}}_y}{x^2 + y^2}$

   (d) $\mathbf{F}(x, y, z) = -A \dfrac{x \widehat{\mathbf{e}}_x + y \widehat{\mathbf{e}}_y + z \widehat{\mathbf{e}}_z}{(x^2 + y^2 + z^2)^{3/2}}$

   (e) $\mathbf{F}(x, y) = A \dfrac{-y \widehat{\mathbf{e}}_x + x \widehat{\mathbf{e}}_y}{x^2 + y^2}$ [*Warning:* Be careful with this one!!]

4. Two objects of equal mass, each having initial speed v but moving in different directions, undergo a completely inelastic collision. After the collision they are moving with speed v'. What was the angle θ between their initial directions? [*Hint:* To simplify your calculations, choose the objects' initial directions to be symmetrically located with respect to the x axis.]

5. A ball of mass m, traveling to the right at speed v, collides head-on with a ball of mass $2m$ that is initially at rest. The collision is elastic. Find the velocities of the two masses after the collision, by two methods:

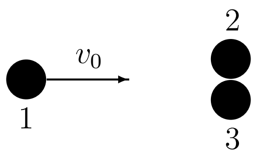
   (a) by working entirely in the laboratory frame;

   (b) by transforming to the center-of-mass frame and then transforming back.

6. Kleppner and Kolenkow, Problem 4.25.

7. Kleppner and Kolenkow, Problem 4.26.

8. A billiard ball (#1) moves to the right with an initial speed v_0. It collides elastically with a pair of identical balls (#2,3) that are initially in contact, as shown below:



(a) In what directions do balls 2 and 3 emerge from the collision? [*Hint:* Draw a careful picture of the situation right at the moment of contact between the three balls, and deduce the direction of the force that ball 1 exerts on each of balls 2 and 3.]

(b) Along what axis does ball 1 emerge from the collision? [*Hint:* Use symmetry. It may not be clear at first whether ball 1 emerges in the + or − direction along this axis; that will come out later, when you solve the equations. For now, just choose a convention as to which direction you consider +.]

(c) Let v_2 and v_3 be the speeds of balls 2 and 3 after the collision. What is the relation between v_2 and v_3? [*Hint:* Use symmetry, or alternatively use conservation of momentum.]

(d) Let v_1 be the velocity with which ball 1 emerges from the collision (according to your convention of what you consider +). Write the relation between v_0, v_1 and v_2 which expresses conservation of momentum.

(e) Write the relation between v_0, v_1 and v_2 which expresses conservation of energy.

(f) Solve these two equations simultaneously to find v_1 and v_2 in terms of v_0. [*Hint:* Square the first equation, and subtract the second one from it. Solve for v_2 in terms of v_1, then plug back in to get everything in terms of v_0.]