MATHEMATICS 1302 (Applied Mathematics 2)
YEAR 2008–2009, TERM 2

PROBLEM SET #2

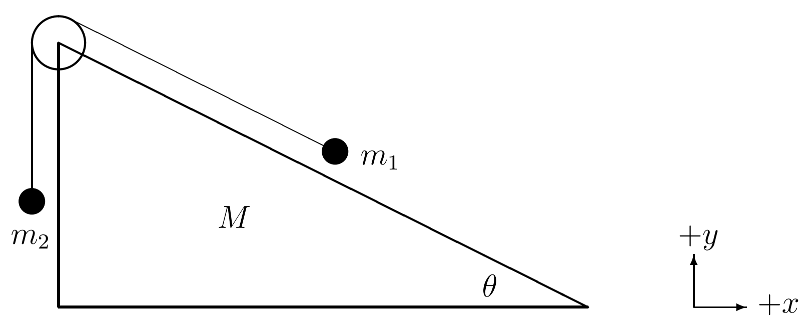
This problem set is due at the *beginning* of problem class on Wednesday 4 February.

Topics: How to set up mechanics problems (conclusion). Conservation of momentum. Conservation of angular momentum. Solvable cases of one-dimensional motion (review). Energy in one-dimensional motion.

Readings:

- Kleppner and Kolenkow, Sections 3.1–3.4 (handout).
- Kleppner and Kolenkow, Sections 6.1–6.3 (handout).
- Handout #5: Solvable cases of one-dimensional motion.

1. In problem 5 of Problem Set 1, you considered a system in which a block of mass m_1 lying on a smooth (frictionless) inclined plane of angle θ is connected by a cord over a massless frictionless pulley to a second block of mass m_2 hanging vertically. In that problem the inclined plane was bolted to the earth, but now I want to change the problem slightly: the inclined plane, which has mass M, rests on a rough floor with coefficient of static friction μ_s.



   (a) Draw single-body diagrams for m_1, m_2 and the inclined plane. Show all the forces that act on each of the bodies, and label them to show their causes.

   (b) Write down the equations of motion for the system, assuming that the inclined plane remains at rest.

   (c) Solve for the accelerations of m_1 and m_2 and the tension in the cord.

   (d) Find the smallest coefficient of static friction μ_s for which the inclined plane will remain at rest.

2. Moe (mass 50 kg) and Joe (mass 100 kg) are ice skaters. They are connected by a massless unstretchable rope of length 10 m. Initially they are at rest, and the rope is fully extended. Assume that the ice is frictionless.

   (a) Moe begins to pull on the rope in such a way that his distance from Joe is reduced at a constant rate of 1 m/sec. Describe their motion during this process: what is Moe's velocity (relative to the earth), and what is Joe's velocity? What happens when Moe and Joe collide?

   (b) For their next act, Joe pushes Moe away in such a way that afterwards their separation is increasing at a constant rate of 2 m/sec. Describe their motion during this process: what is Moe's velocity (relative to the earth), and what is Joe's velocity? What happens when the rope becomes taut?

3. The following problem is taken from Halliday and Resnick, *Physics*:

   > Ricardo (mass 80 kg) and Carmelita are enjoying Lake Merced at dusk in a 30-kg canoe. When the canoe is at rest in the placid water they exchange seats, which are 3 m apart and are symmetrically located with respect to the canoe's center. Ricardo notices that the canoe moved 0.4 m relative to a submerged log, and calculates Carmelita's mass, which she has declined to tell him. What is it?

   (Please excuse the sexist implications of this problem. The physics is interesting, in any event.)

   By the way, this problem actually has *two* solutions, but Messrs. Halliday and Resnick overlooked one of them because of yet another sexist assumption that they made. Can you figure out what their mistake was?

4. Consider a closed jar of mass m_J containing a mass m_A of air and a fly of mass m_F, the whole thing standing on a scale in vacuum (so that we may ignore any buoyant force due to the atmosphere) in the Earth's gravitational field. Of course if the fly is on the wall of the jar, the scale reads the total weight $W = (m_J + m_A + m_F)g$. But suppose the fly is flying around the jar? Under what conditions will the scale read exactly W? Under what conditions will it read something different from W? [*Hint*: Use the center-of-mass theorem.]

5. David and Goliath are pushing an automobile of mass m, which is initially at rest, along a frictionless road. Goliath, strong but out of shape, applies a *forward* push of magnitude $F_G(t) = F_0 e^{-t/\tau}$. David, weaker but with more endurance, applies a *backward* push of magnitude $F_D(t) = \frac{1}{2} F_0 e^{-t/2\tau}$. What net displacement do they produce as $t \to +\infty$?

6. A particle is dropped and falls vertically through air. Assume that the air resistance is quadratic: $F = -\beta mv|v|$. (By the way, why do I write pedantically $F = -\beta v|v|$ rather than simply $F = -\beta v^2$?)

   (a) Find the limiting speed of fall. (Do this by elementary means, i.e. *without* solving any differential equation.)

   (b) Assume that the object is dropped from rest at $t = 0$. Find the velocity as a function of time, by solving the differential equation.

7. Consider a particle with initial velocity v_0, subject only to the retarding force $F = -kv|v|^{n-1}$ with $k, n > 0$. Find $v(t)$ and $x(t)$, and investigate the behavior of v and x as $t \to +\infty$. There will be three cases:

   (a) For *small* n, the particle comes to rest after a finite time, and thus has travelled a finite distance.

   (b) For *intermediate* n, the particle comes to rest only asymptotically as $t \to +\infty$, but the distance it travels as $t \to +\infty$ is finite.

   (c) For *large* n, the particle comes to rest only asymptotically as $t \to +\infty$, and it travels an infinite distance as $t \to +\infty$.

   Prove this scenario, find the values of n that form the dividing lines between these three cases, and put the dividing-line values of n into the correct cases.

8. A particle moves along the x axis, subject to the force $F = -kx|x|^{n-2}$ where $k, n > 0$.

   (a) Find the associated potential energy $V(x)$.

   (b) Use energy considerations to show that the motion will be some form of oscillation between the positions $x = A$ and $x = -A$, for some constant A (called the *amplitude of oscillation*). Find the relation between the amplitude of oscillation A and the total energy E.

   (c) Use an argument based on dimensional analysis to show that the period of oscillation τ is proportional to a certain power of the amplitude of oscillation A, namely $\tau = \alpha A^r$, and find the power r in terms of n. Explain why the constant of proportionality α cannot be determined by dimensional analysis alone. [*Hint:* The period of oscillation τ can only depend on the amplitude of oscillation A and on the constants k and n that characterize the force. So find the dimensions of τ, A, k and n — each will be of the form $L^a T^b M^c$, where L denotes length, T denotes time, M denotes mass, and a, b, c are some exponents. Then ask yourself: In what ways can I combine the quantities A, k and n so that the result has the dimensions of τ?]

   (d) Now solve the problem from scratch using energy methods applied to the differential equation of motion, verifying that r is as found previously and determining α in terms of a *dimensionless* integral (i.e., one whose dimensions are $L^0 T^0 M^0$).

[*Optional:* Evaluate this integral in terms of beta or gamma functions, using your favorite table of integrals.] Please note that you can do part (d) even if you did not succeed in doing part (c).