MATH1302, Question Sheet 8

Questions 1, 2 and 5 to be handed in Tuesday 18 March before the lecture.

Qu 1 *Sketch* the profile of the wave $y(x,t) = e^{-(x-t)^2}$ for $t = 0, 1, 2, 3, 4$. Sketch also the profile $y(x,t) = y_1(x,t) + y_2(x,t)$ of the superposed waves $y_1(x,t) = e^{-(x-t)^2}$, $y_2(x,t) = e^{-(x+t)^2}$, showing resulting profile for the sequence $t = -4, -3, -2, -1, 0, 1, 2, 3, 4$.

Qu 2 Explain the difference between progressive and standing waves. Show how the standing wave $y(x,t) = A\cos(\alpha x)\sin(\beta t)$ can be produced from two progressive waves. Find the amplitudes, frequencies, wavelengths and speeds of these two progressive waves.

Qu 3 Consider the superposition of two waves $y_1(x,t) = A\cos(k_1(x - c_1 t))$ and $y_2(x,t) = A\cos(k_2(x - c_2 t))$. Find and simplify an expression for the superposition $y(x,t) = y_1(x,t) + y_2(x,t)$. Describe carefully the progression of the wave $y(x,t)$ when $|k_1 - k_2|$ is small in the cases (i) $c_1 = c_2$ and (ii) $c_1 > c_2$.

Qu 4 For the previous question, find the points (x,t) (for $x \in \mathbb{R}$, $t \geq 0$) where the amplitude of the superposition wave $y(x,t)$ is $2A$.

Qu 5 A harp string length ℓ is stretched in a straight line between two points. The plucked string vibrates according to the wave equation

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 y}{\partial x^2}, \tag{1}$$

where T is the tension (treated as constant) in the string and $\rho > 0$ a constant. Show that (1) has standing wave solutions of the form $y(x,t) = A\sin kx \sin(\omega t + \delta)$ and find expressions for k and ω. What does ω represent, and how does it vary as the tension T is increased?

A testing robot can pull the string into any initial profile $y(x,0) = Y(x)$. Verify that for each A_n the function

$$y(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{\ell}\right)\cos\left(\frac{n\pi ct}{\ell}\right), \tag{2}$$

where $c = \sqrt{T/\rho}$, satisfies both the wave equation (1) and the boundary conditions $y(0,t) = 0$ and $y(\ell,0) = 0$ and the initial condition that the string is released from rest at $t = 0$.

Prove (by multiplying (2) by $\sin\left(\frac{n\pi x}{\ell}\right)$ and integrating) that $A_n = \dfrac{2}{\ell}\displaystyle\int_0^{\ell} Y(x)\sin\left(\frac{n\pi x}{\ell}\right)\,dx$. What profile $Y(x)$ should the robot pull the string to in order to obtain just the first harmonic?