MATH1302, Question Sheet 7

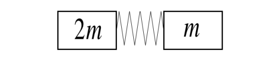
Questions 1, 4 and 5 to be handed in Tuesday 11 March before the lecture.

Qu 1 Consider the equations for the two-mass, three-spring system (see example in lecture notes) with $\sigma = k/m > 0$ where k is the stiffness of the springs, m the mass of each of the two masses, and x, y their displacements from equilibrium

$$\begin{aligned} \ddot{x} &= -2\sigma x + \sigma y \\ \ddot{y} &= \sigma x - 2\sigma y \end{aligned} \tag{1}$$

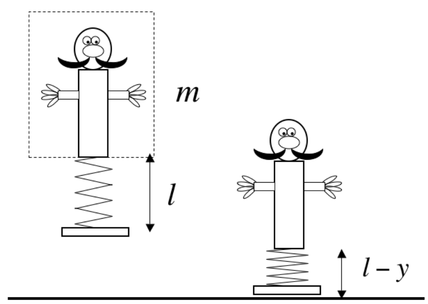
By introducing new variables $X = x + y$ and $Y = x - y$, find the general solution of (1).

Qu 2



Consider two blocks of mass m and mass $2m$ on a frictionless table connected by a spring with stiffness k and unstretched length ℓ. They are compressed together so that the string has length $\ell/2$. Find the maximum speeds of the two blocks after they are released.

Qu 3



Zebedee, the bouncing character from the Magic Roundabout, consists of a upper cylindrical body of mass m (body inside dashed box above), a light spring of unstretched length ℓ and a light flat circular base. He is bouncing periodically and vertically on the spot under gravity, and so that the base reaches a maximum height h. Show that the period of his bouncing is

$$2\sqrt{\frac{2h}{g}} + 2\sqrt{\frac{m}{k}}\left\{\pi - \tan^{-1}\sqrt{\frac{2hk}{mg}}\right\}.$$

(Assume that the spring is never fully compressed during the motion, and that the collision of the light plate with the ground is inelastic.)

Qu 4 A steel ball of mass m is suspended from a fixed beam using an elastic string of natural length ℓ and modulus of elasticity λ. Below the ball is a solenoid which produces a magnetic field that provides a magnetic pull

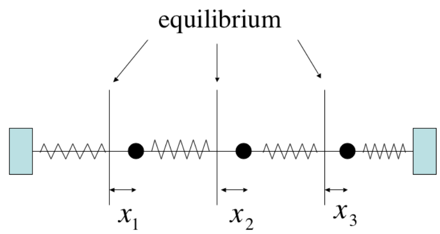
$$F(t) = A(1 + \cos(\omega t)), \quad (A, \omega > 0)$$

on the steel ball. Initially the ball hangs in equilibrium. The solenoid is then turned on. Show that while the string remains taught the displacement from equilibrium y of the ball satisfies

$$\ddot{y} + \frac{\lambda}{m} y = \frac{A}{m}(1 + \cos(\omega t)),$$

and when $\omega = \sqrt{\lambda/m}$ find the ball's displacement while the string remains taut. (Assume that the ball does not collide with the solenoid and that motion is entirely vertical.)

Qu 5 Consider the 3 mass generalisation of Qu 1. The springs and mass are identical with stiffness k and m respectively.



Show that motion of the system about equilibrium is given by

$$\ddot{\underline{x}} = -A\underline{x}, \tag{2}$$

where $\underline{x} = (x_1, x_2, x_3)^T$ is the vector of mass displacements from equilibrium, and

$$A = \begin{pmatrix} 2\sigma & -\sigma & 0 \\ -\sigma & 2\sigma & -\sigma \\ 0 & -\sigma & 2\sigma \end{pmatrix},$$

where $\sigma = k/m$.

Show that the general solution for the mass displacements is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \alpha \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \cos((2\sigma)^{\frac{1}{2}} t + \delta_1)$$

$$+\beta \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} \cos((2\sigma - \sqrt{2}\sigma)^{\frac{1}{2}} t + \delta_2) + \gamma \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} \cos((2\sigma + \sqrt{2}\sigma)^{\frac{1}{2}} t + \delta_2).$$

where α, β, γ and $\delta_1, \delta_2, \delta_3$ are constants.

Qu 6 An oscillator with equation $\ddot{y} + y = \epsilon \dot{y}^3$ is operating with constant $\epsilon \ll 1$. (a) Verify that the approximation $y = y_1(t) + \epsilon y_2(t)$ gives $\ddot{y}_1 + y_1 = 0$ and $\ddot{y}_2 + y_2 = \dot{y}_1^3$. If $y_1 = \cos t$, (b) show that the y_2 equation has a first and third harmonic[1] on its right-hand side. (c) Solve for y_2 given that $y_2(0) = 0$ and $\dot{y}_2(0) = 0$, and deduce that the approximation becomes invaid for large t.

[1] The kth harmonic here is a term of the form $\cos(kt + \delta)$ for some δ.