MATH1302, Question Sheet 4

Questions 3, 5 and 6 to be handed in Tuesday 19 February (after reading week) before the lecture.

Qu 1 A surface is given parametrically by

$$\underline{r} = ((\phi + \sin\phi)\cos\theta, (\phi + \sin\phi)\sin\theta, (1 - \cos\phi)), \quad \phi \in (-\pi, \pi), \, \theta \in [0, 2\pi).$$

Find the equations for the surface in cylindrical coordinates ρ, θ, z (in terms of the parameter ϕ) and sketch the surface.

Qu 2 A heavy bead slides down a frictionless helical wire whose shape is given by $x = a\cos\theta, y = a\sin\theta, z = -b\theta$ for $a, b > 0$. If the bead starts at $z = 0$, find its speed when it has rotated about the $z-$axis n times.

Qu 3 (Conical pendulum) A pendulum consists of a light string length a fixed at O and with a mass m attached to the free end A. The string is held so that OA makes an angle α to the downward vertical and set in motion so that horizontal circular motion of the mass m occurs. What is the period of this circular motion?

Qu 4 A heavy particle moves inside a frictionless hollow sphere radius a. Show that its motion satisfies

$$\frac{1}{2}m\left(\frac{a^2\dot{z}^2}{a^2 - z^2} + \frac{h^2}{a^2 - z^2}\right) + mgz = \text{constant},$$

where z is measured upwards from the centre of the sphere, and h is a constant. The particle is held on the inner surface of the sphere at $z = -a/2$ and given a horizontal velocity U. Find the maximum and minimum height that the particle reaches during the ensuing motion.

Qu 5 (M13B Exam 2005, Q 4) A funnel has a frictionless surface, given by $z = b(b/\rho)^n$ in cylindrical coordinates ρ, θ, z with $z > 0$ vertically downwards, where $b > 0, n > 1$ are constants. A heavy particle of mass m is projected horizontally with speed U along the inner surface at the level $z = b$.

Show that $\rho^2\dot{\theta} = Ub$ and

$$\left(1 + n^2\left(\frac{b}{\rho}\right)^{2n+2}\right)\dot{\rho}^2 + \left(\frac{Ub}{\rho}\right)^2 - 2gb\left(\frac{b}{\rho}\right)^n = U^2 - 2gb,$$

where g is accleration due to gravity.

If the particle is found to be moving horizontally again at $z = 2^n b$, prove that $3U^2 = (2^n - 1)2gb$.

Qu 6 A particle P mass m on a frictionless horizontal table is joined by a light string length l that passes through a hole O in the table and supports at its other end a second particle Q mass m' that hangs below the table under gravity. Initially the particle P is projected with speed V perpendicular to OP, where $|OP| = a < l$. Assuming that thereafter the string remains taut, show that if $r = |OP|$ and $z = |OQ|$, then

$$m(\ddot{r} - r\dot{\theta}^2) = m'(\ddot{z} - g), \quad r^2\dot{\theta} = h,$$

where h is a constant which you should find.

Show that horizontal circular motion of radius a is possible if $V = \sqrt{\frac{m'ga}{m}}$. Show also that for small perturbations x from this circular motion that preserve h,

$$\ddot{x} = -\frac{3mh^2}{(m + m')a^4}x,$$

to first order in x.