MATH1302, Question Sheet 3

Questions 2, 3, 5 to be handed in Tuesday 29 January before the lecture.

Qu 1 A particle P mass m moves in the plane under the action of an attractive central force $\underline{F}$. Show that its angular momentum in cartesian coordinates (x, y) is $m(x\dot{y} - y\dot{x})$. Write down expressions in the cartesian coordinates for the accelerations $\ddot{x}$ and $\ddot{y}$. Hence show that the particle's angular momentum is conserved.

Qu 2 (The orbits of planets about the sun are planar)
Consider the three dimensional motion of a planet about the sun. Take the sun as the origin and let $\underline{r} = (x(t), y(t), z(t))$ be the position vector of a planet at time t. Show that the angular momentum $\underline{L} = m\underline{r} \times \dot{\underline{r}}$ of the planet about the sun is constant. Show also that the planet moves in a plane containing the sun. (Neglect the gravitational effects of other planets.)

Qu 3 State the formulae for radial and transverse acceleration of a particle in polar coordinates r, θ. For the motion of a particle position vector $\underline{r}$ under an attractive central force of magnitude $F(r, \theta)$ derive the equation for $u = 1/r$:

$$\frac{d^2u}{d\theta^2} + u = \frac{F(1/u, \theta)}{mh^2u^2}.$$

What is the constant h?

A particle P of unit mass moves with position vector $\underline{r}$ under a central force

$$\underline{F} = -\frac{k(2 + \cos 2\theta)^{-\frac{3}{2}}}{r^3}\underline{r}.$$

Show that the particle moves in orbits whose cartesian equation are

$$Ax + By - 1 = \frac{k}{3h^2}\sqrt{x^2 - y^2},$$

where h, A, B are constants. (Hint: In the $u-$equation for motion look for a particular integral proportional to $\sqrt{2 + \cos 2\theta}$ and then convert into cartesian coordinates.)

Qu 4 A particle with position vector $\underline{r}$ moves under a force

$$\underline{F} = \frac{k}{r^4}\underline{r}$$

per unit mass, where $r = |\underline{r}|$ and k is *any* real number. The particle starts at $r = a$ with radial velocity $U > 0$ and tangential velocity $V > 0$. Find the equation for the particle's path in polar coordinates and discuss whether the particle motion is bounded or unbounded, being careful to consider the outcomes for different k.

Qu 5 A meteorite is approaching the Earth and is first detected at a very large distance away (i.e. $r = \infty$) when it is moving with speed U. If the meteorite were to continue in the absence of the Earth's gravitational pull it would pass the Earth at a minimum distance d. If G is the gravitational constant and M the mass of the Earth, find the nearest that the meteorite comes to the Earth when subject to its gravitational pull.

Qu 6 (Tricky!) A satellite orbits the Earth under its gravitational pull and in an elliptical orbit with semi-major axes a and semi-minor axes b. Show that if orbital period is T then

$$\frac{1}{T}\int_0^T \frac{1}{r}\,dt = \frac{1}{a}.$$

(Hint: Change variables to θ, use conservation of angular momentum and the polar equation for an ellipse. You may quote $\ell = a(1-e^2)$ and $b = a\sqrt{1-e^2}$ and that the area of an ellipse is πab). Deduce that if v is the satellite's speed,

$$\frac{1}{T}\int_0^T v^2\,dt = \frac{GM}{a},$$

where G is the gravitational constant and M the mass of the Earth. (Use conservation of energy.)