MATH1302, Question Sheet 2

Questions 2,4,5 to be handed in Tuesday 29 January before the lecture.

Qu 1 Show that the radius of curvature ρ of a plane curve can be written as

$$\rho = \frac{\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}{\left|\frac{d^2y}{dx^2}\right|}.$$

Show also that for a curve given parametrically by $x = x(t), y = y(t)$,

$$\rho = \frac{(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}}}{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}.$$

Hence show that the radius of curvature of an ellipse with semi-major axis a, semi-minor axis b is

$$\rho = \frac{1}{ab}\left(b^2 + \left(\frac{a^2}{b^2} - 1\right)y^2\right)^{\frac{3}{2}}$$

Qu 2 Sketch the curve $y(x) = \log(1 + \cos x)$ on the interval $(0, \pi)$. Taking the signed arclength $s = 0$ at $x = 0$, show that

$$x(s) = 4\tan^{-1}\left(\tanh\frac{s}{4}\right),$$

and find an expression for $y(s)$. Find also ψ as a function of s.

Qu 3 A heavy bead moves on a smooth and strictly convex[1] curve $y = y(x)$ for $x \in \mathbb{R}$. The curve $y(x)$ also satisfies $y(x) \to \infty$ as $|x| \to \infty$ and has its (unique) minimum at $x = 0$. Show that if ρ_0 is the radius of curvature at $x = 0$ then the period of small oscillations about the minimum is approximately $\dfrac{2\pi}{\omega}$ where $\omega^2 = \frac{g}{\rho_0}$. (Hint: do a Maclaurin expansion of $\rho(s)$).

Qu 4 A heavy particle B rests at the lowest point A of the inside of a fixed smooth spherical shell centre O and radius a. The particle is then struck so that its initial speed is U. Find the reaction force as a function of the angle θ that the line OA makes with OB. What are the conditions on U that the particle never leaves the inner surface of the sphere?

Qu 5 A curve C is given parametrically by $x = \theta + \sin\theta$, $y = 1 - \cos\theta$ on the interval $\theta \in (-\pi, \pi)$. Show that if s is the signed arclength from the origin and ψ the angle between the tangent and the positive $x-$axis then $s = 4\sin\psi$.

A smooth bowl is made from the surface of revolution formed by rotating the above curve C about the vertical axis through the origin. A heavy particle initially sits at the bottom of the bowl. It is struck so that its initial speed is U. Find the period of the ensuing oscillation.

Qu 6 A particle moves at a constant speed u on a plane curve γ, and the particle's component of acceleration along the $y-$axis is constant. Show that γ is a cycloid.

[1] A curve $y = y(x)$ is strictly convex if $y''(x) > 0$ for all $x \in \mathbb{R}$