MATHEMATICS 0054 (Analytical Dynamics)
YEAR 2021–2022, TERM 2

HANDOUT #8: COUPLED OSCILLATIONS AND NORMAL MODES

1 Review of simple harmonic oscillator

In MATH 0008/0009[1] you studied the *simple harmonic oscillator*: this is the name given to any physical system (be it mechanical, electrical or some other kind) with one degree of freedom (i.e. one dependent variable x) satisfying the equation of motion

$$m\ddot{x} \;=\; -kx \;, \tag{1}$$

where m and k are positive constants (and the dot $\dot{}$ denotes d/dt as usual). For instance, if we have a particle of mass m attached to a spring of spring constant k (with the other end of the spring attached to a fixed wall), then the force on the particle is $F = -kx$ where x is the particle's position (with $x = 0$ taken to be the equilibrium point of the spring), so Newton's Second Law $F = ma$ is indeed (1).

Let us review briefly the solution of the harmonic-oscillator equation (1). Since this is a one-dimensional problem with a position-dependent force, it can be solved by the energy method, with potential energy $U(x) = \frac{1}{2}kx^2$. But a simpler method is to recognize that (1) is a homogeneous linear differential equation with constant coefficients, so its solutions can be written (except in certain degenerate cases) as linear combinations of suitably chosen exponentials, which we can write either as $x(t) = e^{\alpha t}$ or as $x(t) = e^{i\omega t}$. Let us use the latter form (which is more convenient for oscillatory systems, because ω will come out to be a real number). So the method is to *guess* a solution of the form

$$x(t) \;=\; e^{i\omega t} \tag{2}$$

and then choose ω so that this indeed solves (1). Inserting $x(t) = e^{i\omega t}$ into (1), we find

$$-m\omega^2 e^{i\omega t} \;=\; -ke^{i\omega t} \;, \tag{3}$$

which is a solution if (and only if) $\omega = \pm\sqrt{k/m}$. We conclude that the general solution of (1) is

$$x(t) \;=\; Ae^{i\omega t} \;+\; Be^{-i\omega t} \tag{4}$$

with $\omega = \sqrt{k/m}$.[2] This can equivalently be written in "sine-cosine" form as

$$x(t) \;=\; C_1 \cos\omega t \;+\; C_2 \sin\omega t \tag{5}$$

or in "amplitude-phase" form as

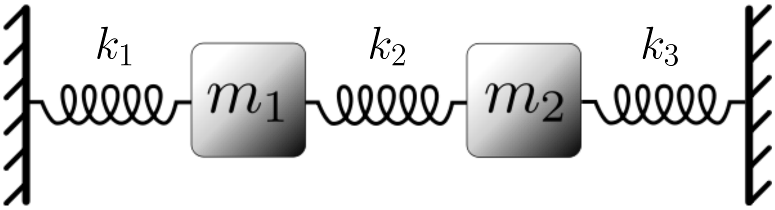
$$x(t) \;=\; C\cos(\omega t + \phi) \;. \tag{6}$$

[1] Formerly MATH 1301/1302.

[2] This is wrong when $\omega = 0$, because then the solutions $e^{i\omega t}$ and $e^{-i\omega t}$ are not linearly independent. In this case the linearly independent solutions are $e^{i\omega t}$ (i.e. 1) and $te^{i\omega t}$ (i.e. t). This is the "degenerate case" I referred to earlier.

2 Coupled oscillations: A simple example

Now let us consider a simple situation with *two* degrees of freedom. Suppose we have two particles, of masses m_1 and m_2, respectively, connected between two walls via three springs of spring constants k_1, k_2, k_3, as follows:



Let us assume for simplicity that the distance between the walls is exactly the sum of the equilibrium lengths of the three springs. And let us measure the positions of the two particles (x_1 and x_2) relative to their equilibrium positions. Then the equations of motion are

$$m_1\ddot{x}_1 = -k_1 x_1 + k_2(x_2 - x_1) \tag{7a}$$
$$m_2\ddot{x}_2 = -k_3 x_2 + k_2(x_1 - x_2) \tag{7b}$$

(You should check this carefully and make sure you understand the *signs* on all four forces.)

Let us now try a solution of the form

$$x_1(t) = A_1 e^{i\omega t} \tag{8a}$$
$$x_2(t) = A_2 e^{i\omega t} \tag{8b}$$

where A_1 and A_2 are constants. Substituting this into (7) yields

$$-m_1\omega^2 A_1 e^{i\omega t} = -(k_1+k_2)A_1 e^{i\omega t} + k_2 A_2 e^{i\omega t} \tag{9a}$$
$$-m_2\omega^2 A_2 e^{i\omega t} = k_2 A_1 e^{i\omega t} - (k_2+k_3)A_2 e^{i\omega t} \tag{9b}$$

Extracting the common factor $e^{i\omega t}$ and moving everything to the right-hand side, we obtain

$$(m_1\omega^2 - k_1 - k_2)A_1 + k_2 A_2 = 0 \tag{10a}$$
$$k_2 A_1 + (m_2\omega^2 - k_2 - k_3)A_2 = 0 \tag{10b}$$

which is most conveniently written in matrix form as

$$\begin{pmatrix} m_1\omega^2 - k_1 - k_2 & k_2 \\ k_2 & m_2\omega^2 - k_2 - k_3 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} . \tag{11}$$

This is a homogeneous linear equation; it has a nonzero solution (i.e. a solution other than $A_1 = A_2 = 0$) if and only if the matrix on the left-hand side is singular, i.e. has a zero determinant. Setting the determinant equal to zero gives a quadratic equation for ω^2, namely

$$(m_1\omega^2 - k_1 - k_2)(m_2\omega^2 - k_2 - k_3) - k_2^2 = 0 , \tag{12}$$

which can be solved by the quadratic formula. Then, for each of the two possible values for ω^2, we can go back to the linear equation (11) and solve for the "eigenvector" $\begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$.

In particular, in the "symmetric case" $k_1 = k_2 = k_3 = k$ and $m_1 = m_2 = m$, the solutions are

$$\omega_1 = \sqrt{k/m} \tag{13a}$$
$$\omega_2 = \sqrt{3k/m} \tag{13b}$$

and the corresponding eigenvectors are

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{14a}$$
$$\mathbf{e}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{14b}$$

The **normal modes** — that is, the solutions of (7) that are pure oscillations at a single frequency — are therefore

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\sqrt{k/m}\, t + \phi_1) \tag{15}$$

and

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(\sqrt{3k/m}\, t + \phi_2)\ . \tag{16}$$

The general solution is a linear combination of these two normal modes. In the first (slower) normal mode, the two particles are oscillating in phase, with the same amplitude; the middle spring therefore exerts no force at all, and the frequency is $\sqrt{k/m}$ as it would be if the middle spring were simply absent. In the second (faster) normal mode, the two particles are oscillating 180° out of phase, with the same amplitude; therefore, each particle feels a force that is $-3k$ times its displacement (why?), and the frequency is $\sqrt{3k/m}$.

This solution can be interpreted in another way. Let us build a matrix N whose columns are the eigenvectors corresponding to the normal modes,

$$N = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\ , \tag{17}$$

and let us make the change of variables

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = N \begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix} \tag{18}$$

or equivalently

$$x_1 = x'_1 + x'_2 \tag{19a}$$
$$x_2 = x'_1 - x'_2 \tag{19b}$$

Then a solution with $x'_1 \neq 0$, $x'_2 = 0$ corresponds to the first normal mode, while a solution with $x'_1 = 0$, $x'_2 \neq 0$ corresponds to the second normal mode. The point is that the change of variables (18)/(19) *decouples* the system (7) [when $m_1 = m_2 = m$]: after a bit of algebra we obtain

$$m\ddot{x}'_1 \;=\; -kx'_1 \tag{20a}$$

$$m\ddot{x}'_2 \;=\; -3kx'_2 \tag{20b}$$

(You should check this!) In other words, by a linear change of variables corresponding to passage to the normal modes, the system (7) of *coupled* harmonic oscillators turns into a system (20) of *decoupled* simple harmonic oscillators, each of which may be solved *separately* by the elementary method reviewed in Section 1.

3 Coupled oscillations: The general case

We can now see how to handle the general case of coupled oscillators with an arbitrary finite number n of degrees of freedom. We will have a system of homogeneous linear constant-coefficient differential equations of the form

$$M\ddot{\mathbf{x}} + K\mathbf{x} \;=\; \mathbf{0} \tag{21}$$

where

- $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ is a column vector of coordinates;
- M (the so-called **mass matrix**) is a symmetric positive-definite $n \times n$ real matrix (usually it will be a diagonal matrix, but it need not be);
- K (the so-called **stiffness matrix**) is a symmetric $n \times n$ real matrix (usually it too will be positive-definite, but it need not be); and
- $\mathbf{0}$ denotes the zero vector.

We then try a solution of the form

$$\mathbf{x}(t) \;=\; \mathbf{e}\, e^{i\omega t} \tag{22}$$

where $\mathbf{e}$ is some fixed vector. This will solve (21) if (and only if)

$$(K - \omega^2 M)\mathbf{e} \;=\; \mathbf{0} \tag{23}$$

or equivalently

$$K\mathbf{e} \;=\; \omega^2 M\mathbf{e}\,. \tag{24}$$

This is a **generalized eigenvalue problem** (it would be the ordinary eigenvalue problem if M were the identity matrix). The **eigenvalues** — that is, the values of $\lambda = \omega^2$ for which there exists a solution $\mathbf{e} \neq \mathbf{0}$ — are the solutions of the nth-degree polynomial equation

$$\det(K - \lambda M) \;=\; 0 \tag{25}$$

[here the polynomial $p(\lambda) = \det(K - \lambda M)$ is called the **characteristic polynomial**]; and then the corresponding vectors $\mathbf{e} \neq \mathbf{0}$ that solve (23)/(24) are the **eigenvectors**. The solution $\mathbf{x}(t) = \mathbf{e}\, e^{i\omega t}$ [or $\mathbf{x}(t) = \mathbf{e} \cos \omega t$] corresponding to such an **eigenpair** $(\lambda, \mathbf{e})$ is a **normal mode**.

So here is what we need to do in any concrete problem: Solve the nth-degree polynomial equation $\det(K - \lambda M) = 0$ to get the n eigenvalues[3]; and then for each eigenvalue, solve the linear system to get the corresponding eigenvector. We expect to get n distinct eigenpairs, i.e. n normal modes.

Is this guaranteed to work? If the n eigenvalues are distinct, then the answer is clearly yes: To each eigenvalue there will correspond *at least one* eigenvector (since the vanishing of the determinant guarantees that the linear system has a nonzero solution); the eigenvectors for different eigenvalues will be linearly independent (that follows from general theory); and since the whole space has dimension only n, it must be that to each eigenvalue there corresponds *exactly one* eigenvector (except of course for trivial rescalings, i.e. multiplying that eigenvector by some nonzero constant). Hence we will get exactly n normal modes.

But what if the n eigenvalues are *not* distinct (i.e. one or more of the roots of the characteristic polynomial are multiple roots)? One might worry that some eigenvalue of multiplicity 2 has only a one-dimensional subspace of eigenvectors, or more generally that some eigenvalue of multiplicity k has an ℓ-dimensional subspace of eigenvectors where $\ell < k$. Can this happen? Well, it *can* happen for matrices in general: for instance, for the matrix $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, the characteristic polynomial is $\det(A - \lambda I) = \lambda^2$, so the eigenvalue 0 has multiplicity 2, but there is only a *one-dimensional* subspace of eigenvectors, namely multiples of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ [you should check this!]. Matrices (like this one) that fail to have a complete basis of eigenvectors are called **defective** or **nondiagonalizable**; they're not pleasant to deal with (as the faintly pejorative terminology suggests!), but they do exist and do occasionally arise in practical problems.

Luckily, it turns out that this disaster *cannot* happen in our present situation, where K is symmetric and M is symmetric positive-definite. In other words, I claim that a generalized eigenvalue problem with a pair of real symmetric matrices, at least one of which is positive-definite, always has a basis of eigenvectors: that is, we can always find eigenvalues $\lambda_1, \ldots, \lambda_n$ and corresponding eigenvectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ such that

(a) $K\mathbf{e}_j = \lambda_j M \mathbf{e}_j$ for $j = 1, \ldots, n$; and

(b) $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is a basis of $\mathbb{R}^n$.

[3]These eigenvalues are not necessarily distinct, since the polynomial equation may have multiple roots. But a polynomial equation of degree n has in any case n roots *counting multiplicity*: that is the Fundamental Theorem of Algebra.

[When $M = I$ this is just the theorem, proved in your linear algebra course, that a real symmetric matrix K has a basis of eigenvectors.] Let us prove this as follows:[4]

Lemma 1 *Every symmetric positive-definite real matrix has a symmetric positive-definite square root. That is, if M is a symmetric positive-definite real $n \times n$ matrix, then there exists another symmetric positive-definite real $n \times n$ matrix, which we shall denote $M^{1/2}$, such that $M^{1/2}M^{1/2} = M$. (In fact this matrix $M^{1/2}$ is unique, but we shall not need this fact.)*

PROOF. Basic linear algebra tells us that any symmetric real matrix can be diagonalized by an orthogonal transformation: that is, there exists a matrix R satisfying $R^{\mathrm{T}}R = RR^{\mathrm{T}} = I$ (that is the definition of "orthogonal matrix") and $R^{\mathrm{T}}MR = D$, where D is a diagonal matrix. In fact, $D = \mathrm{diag}(m_1, \ldots, m_n)$, where $m_1, \ldots, m_n$ are the eigenvalues of M. In our case, the matrix M is positive-definite, so all the eigenvalues $m_1, \ldots, m_n$ are > 0; in particular, they have square roots. We can therefore define the matrix $D^{1/2} = \mathrm{diag}(m_1^{1/2}, \ldots, m_n^{1/2})$, which manifestly satisfies $D^{1/2}D^{1/2} = D$.

Now define $M^{1/2} = RD^{1/2}R^{\mathrm{T}}$. We have

$$\begin{aligned} M^{1/2}M^{1/2} &= RD^{1/2}R^{\mathrm{T}}RD^{1/2}R^{\mathrm{T}} \\ &= RD^{1/2}D^{1/2}R^{\mathrm{T}} \\ &= RDR^{\mathrm{T}} \\ &= R(R^{\mathrm{T}}MR)R^{\mathrm{T}} \\ &= M \end{aligned}$$

where we used $R^{\mathrm{T}}R = I$, then $D^{1/2}D^{1/2} = D$, then $D = R^{\mathrm{T}}MR$, and finally $RR^{\mathrm{T}} = I$. (You should check this carefully!)

I leave it to you to verify that $M^{1/2} = RD^{1/2}R^{\mathrm{T}}$ is symmetric.

Finally, since $M^{1/2} = RD^{1/2}R^{\mathrm{T}}$ with $D^{1/2}$ positive-definite and R nonsingular, it follows that $M^{1/2}$ is positive-definite as well. (You should go back to the definition of "positive-definite matrix" and verify this assertion too.) $\square$

Let us now show that a pair of real quadratic forms, one of which is positive-definite, can be simultaneously diagonalized:

Proposition 2 *Let M and K be a real symmetric $n \times n$ matrices, with M positive-definite. Then there exists a nonsingular real $n \times n$ matrix N such that $N^{\mathrm{T}}MN = I$ (the identity matrix) and $N^{\mathrm{T}}KN = \Lambda$, where Λ is a diagonal matrix.*

PROOF. Let $M^{1/2}$ be the symmetric positive-definite square root of M whose existence is guaranteed by the Lemma. Since $M^{1/2}$ is positive-definite, it is invertible. Then $L = (M^{1/2})^{-1}K(M^{1/2})^{-1}$ is a real symmetric matrix, so it can be diagonalized by an orthogonal transformation: that is, there exists a matrix R satisfying $R^{\mathrm{T}}R = RR^{\mathrm{T}} = I$ and $R^{\mathrm{T}}LR = \Lambda$,

[4] Of course, this general proof is not needed for practical problems; rather, its value is to guarantee in advance that a complete solution will always be found.

where Λ is a diagonal matrix. Now define $N = (M^{1/2})^{-1}R$. We have $N^{\mathrm{T}} = R^{\mathrm{T}}(M^{1/2})^{-1}$ (why?). Then $N^{\mathrm{T}}KN = \Lambda$ by construction (why?), and

$$\begin{aligned} N^{\mathrm{T}}MN &= R^{\mathrm{T}}(M^{1/2})^{-1}M(M^{1/2})^{-1}R \\ &= R^{\mathrm{T}}(M^{1/2})^{-1}M^{1/2}M^{1/2}(M^{1/2})^{-1}R \\ &= R^{\mathrm{T}}R \\ &= I\,. \end{aligned}$$

□

Corollary 3 *Let M and K be a real symmetric $n \times n$ matrices, with M positive-definite. Then there exist real numbers $\lambda_1, \ldots, \lambda_n$ and vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n \in \mathbb{R}^n$ such that*

(a) $K\mathbf{e}_j = \lambda_j M\mathbf{e}_j$ for $j = 1, \ldots, n$; and

(b) $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is a basis of $\mathbb{R}^n$.

Proof. Let N be the matrix whose existence is guaranteed by the Proposition, and let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be its columns. Since N is nonsingular, its columns are linearly independent, hence form a basis of $\mathbb{R}^n$. Obviously N^{T} is also nonsingular, hence invertible, and the Proposition tells us that

$$\begin{aligned} MN &= (N^{\mathrm{T}})^{-1} \\ KN &= (N^{\mathrm{T}})^{-1}\Lambda \end{aligned}$$

(why?). Here Λ is a diagonal matrix; let $\lambda_1, \ldots, \lambda_n$ be its diagonal entries. Now, the jth column of MN is $M\mathbf{e}_j$ (why?), so the jth column of $(N^{\mathrm{T}})^{-1}$ is $M\mathbf{e}_j$. It follows that the jth column of $(N^{\mathrm{T}})^{-1}\Lambda$ is $\lambda_j M\mathbf{e}_j$ (why?). Since the jth column of KN is $K\mathbf{e}_j$, this proves that $K\mathbf{e}_j = \lambda_j M\mathbf{e}_j$. □

This manipulation of matrices is quick but perhaps a bit abstract. Here is a more direct proof of the Corollary that analyzes directly the generalized eigenvalue problem:

Alternate proof of Corollary 3. Let $M^{1/2}$ be the symmetric positive-definite square root of M whose existence is guaranteed by the Lemma. We can then rewrite

$$K\mathbf{e} = \lambda M\mathbf{e} \tag{26}$$

as

$$K\mathbf{e} = \lambda M^{1/2}M^{1/2}\mathbf{e}\,. \tag{27}$$

Defining $\mathbf{f} = M^{1/2}\mathbf{e}$, we have $\mathbf{e} = (M^{1/2})^{-1}\mathbf{f}$ [note that $M^{1/2}$ is invertible because it is positive-definite] and hence the equation can be rewritten as

$$K(M^{1/2})^{-1}\mathbf{f} = \lambda M^{1/2}\mathbf{f}\,. \tag{28}$$

And we can left-multiply both sides by $(M^{1/2})^{-1}$ to obtain

$$(M^{1/2})^{-1}K(M^{1/2})^{-1}\mathbf{f} \;=\; \lambda\mathbf{f} \tag{29}$$

[note that this operation is reversible because $(M^{1/2})^{-1}$ is invertible]. So we now have an *ordinary* eigenvalue problem for the symmetric real matrix $(M^{1/2})^{-1}K(M^{1/2})^{-1}$. This matrix has eigenvalues $\lambda_1, \ldots, \lambda_n$ and a corresponding basis of linearly independent eigenvectors $\mathbf{f}_1, \ldots, \mathbf{f}_n$. Defining $\mathbf{e}_j = (M^{1/2})^{-1}\mathbf{f}_j$, a simple calculation shows that

$$K\mathbf{e}_j \;=\; \lambda_j M\mathbf{e}_j \qquad \text{for } j = 1, \ldots, n\;. \tag{30}$$

And $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is a basis of $\mathbb{R}^n$ because $\{\mathbf{f}_1, \ldots, \mathbf{f}_n\}$ is a basis of $\mathbb{R}^n$ and the matrix $(M^{1/2})^{-1}$ is nonsingular. $\square$

4 Another example: n masses with springs

Now let us generalize the example of Section 2 by considering a chain of n particles, each of mass m, joined by $n+1$ ideal springs, each of spring constant k, between a pair of walls. Then the equations of motion are

$$\begin{aligned}
m\ddot{x}_1 &= -kx_1 + k(x_2 - x_1) && \text{(31a)}\\
m\ddot{x}_2 &= k(x_1 - x_2) + k(x_3 - x_2) && \text{(31b)}\\
m\ddot{x}_3 &= k(x_2 - x_3) + k(x_4 - x_3) && \text{(31c)}\\
&\;\;\vdots\\
m\ddot{x}_{n-1} &= k(x_{n-2} - x_{n-1}) + k(x_n - x_{n-1}) && \text{(31d)}\\
m\ddot{x}_n &= k(x_{n-1} - x_n) - kx_n && \text{(31e)}
\end{aligned}$$

(You should check this carefully and make sure you understand this, including all the signs!) This system of differential equations can be written compactly in the form

$$M\ddot{\mathbf{x}} + K\mathbf{x} \;=\; \mathbf{0} \tag{32}$$

where $M = mI$ and the matrix K has entries $2k$ on the diagonal and $-k$ just above and below the diagonal (and zero entries everywhere else): that is, $K = kL$ where

$$L \;=\; \begin{pmatrix}
2 & -1 & & & & \\
-1 & 2 & -1 & & & \\
 & -1 & 2 & -1 & & \\
 & & \ddots & \ddots & \ddots & \\
 & & & -1 & 2 & -1 \\
 & & & & -1 & 2
\end{pmatrix} \tag{33}$$

is called the **one-dimensional discrete Laplace matrix** (with Dirichlet boundary conditions at the endpoints).[5] The eigenvalues $\lambda = \omega^2$ of our generalized eigenvalue problem are simply k/m times the eigenvalues of the matrix L.

How can we find the eigenvalues and eigenvectors of L? This is not so obvious; it requires a bit of cleverness. Let us start by observing that the equation $L\mathbf{f} = \mu\mathbf{f}$ for the eigenvector $\mathbf{f} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}$ can be written in the simple form

$$-f_{s-1} + 2f_s - f_{s+1} \;=\; \mu f_s \qquad \text{for } s = 1, \ldots, n \tag{34}$$

if we simply define $f_0 = 0$ and $f_{n+1} = 0$. (Why?) Now this is a linear constant-coefficient difference equation; and by analogy with linear constant-coefficient *differential* equations, we might expect the solutions of (34) to be linear combinations of (complex) exponentials, e.g.

$$f_s \;=\; e^{i\alpha s} \tag{35}$$

where we can always take $-\pi < \alpha \le \pi$ (why?). Plugging the guess (35) into (34), we see that this guess indeed solves (34) provided that α and μ are related by

$$2 - 2\cos\alpha \;=\; \mu \; . \tag{36}$$

(You should check this carefully!) In particular we must have $0 \le \mu \le 4$. Note that to each allowed value of μ there corresponds a *pair* of allowed values of α — namely, a value $\alpha > 0$ and its negative — because cos is an even function.[6] So any linear combination of the two solutions $e^{i\alpha s}$ and $e^{-i\alpha s}$ is also a solution for the given value of μ; in particular, any linear combination of $\sin(\alpha s)$ and $\cos(\alpha s)$ is a solution.

But we are not done yet: we have solved the difference equation (34), but we have not yet dealt with the "boundary conditions" $f_0 = 0$ and $f_{n+1} = 0$. The condition $f_0 = 0$ can be satisfied simply by choosing the solution

$$f_s \;=\; \sin(\alpha s) \tag{37}$$

(why?). And the condition $f_{n+1} = 0$ can then be satisfied by making sure that $(n+1)\alpha$ is a multiple of π, i.e.

$$\alpha \;=\; \frac{j\pi}{n+1} \qquad \text{for some integer } j \; . \tag{38}$$

[5] L is called the discrete Laplace matrix because it is the discrete analogue of the Laplacian operator $-d^2/dx^2$ in one dimension or $-\dfrac{\partial^2}{\partial x_1^2} - \cdots - \dfrac{\partial^2}{\partial x_n^2}$ in n dimensions. Indeed, if you were to try to solve on the computer a differential equation involving the operator $-d^2/dx^2$, you would probably discretize space (i.e. replace continuous space by a mesh of closely spaced points) and replace the operator $-d^2/dx^2$ by the matrix L or something similar.

[6] There are two exceptions: $\mu = 0$ corresponds only to $\alpha = 0$, and $\mu = 4$ corresponds only to $\alpha = \pi$.

We can't take $j = 0$, because that would make $\mathbf{f}$ identically zero; but we can take any integer j from 1 up to n. We have thus obtained eigenvectors $\mathbf{f}_1, \ldots, \mathbf{f}_n$ for the matrix L, given by

$$(\mathbf{f}_j)_s \;=\; \sin\!\left(\frac{\pi j s}{n+1}\right) , \tag{39}$$

with corresponding eigenvalues

$$\mu_j \;=\; 2 - 2\cos\!\left(\frac{\pi j}{n+1}\right) \tag{40a}$$

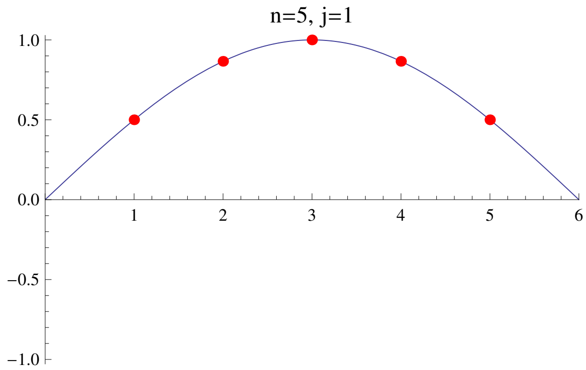
$$\;=\; 4\sin^2\!\left(\frac{\pi j}{2(n+1)}\right) . \tag{40b}$$

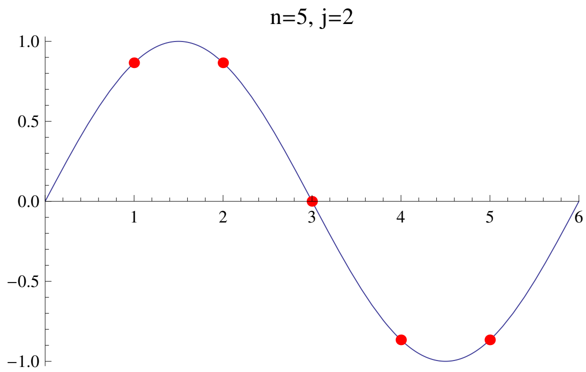
Since there are n of these and they are linearly independent (they must be because the values μ_j are all different!), we conclude that we have found a complete set of eigenvectors for the matrix L.[7]

Physically, these eigenvectors are **standing waves**. To see this, let us make some plots for $n = 5$ and $j = 1, 2, 3, 4, 5$. For each value of j, we first plot the function

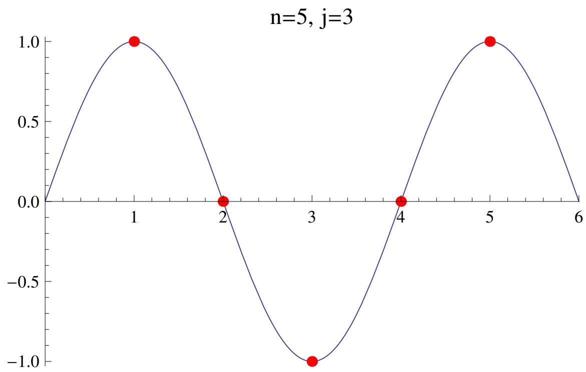
$$f_j(s) \;=\; \sin\!\left(\frac{\pi j s}{n+1}\right) \tag{41}$$

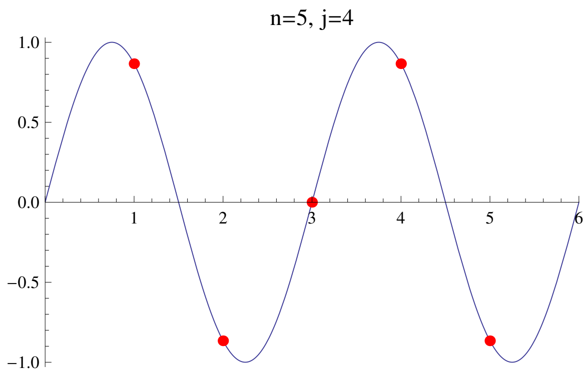
for *real* values of s in the interval $0 \le s \le n+1$ (this shows most clearly the "standing wave"); then we indicate the points corresponding to $s = 1, 2, \ldots, n$, which are the entries in the eigenvector $\mathbf{f}_j$.

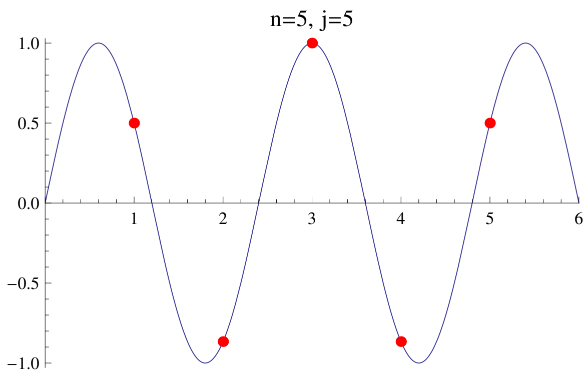




[7]It follows, in particular, that nothing new is obtained by going outside the range $1 \le j \le n$. For instance, $j = n+1$ again yields the zero function, $j = n+2$ yields a multiple of what $j = n$ yields, $j = n+3$ yields a multiple of what $j = n-1$ yields, and so forth.







5 Transverse oscillations of a loaded string

Let us now look at one last example: We consider a (massless) string of length $L = (n+1)d$, tied down at the two ends $x = 0$ and $x = L$ and held at tension T, on which we attach n particles, each of mass m, at the locations $x = d, 2d, 3d, \ldots, nd$. We then consider the *transverse* oscillations of this "loaded string". More precisely, we consider the *small* transverse oscillations: that is, we make the approximation that the transverse displacements are small, and we keep only those terms that are *linear* in those transverse displacements, dropping all terms that are of quadratic or higher order.

So let y_i be the transverse displacement of the ith particle ($i = 1, 2, \ldots, n$); and let us define $y_0 = y_{n+1} = 0$ to indicate where the ends of the string are tied down. If the displacements y_i are small, then the tension in the string is approximately constant and equal to its value at equilibrium, i.e. T. The element of string connecting particle i to particle $i+1$ is inclined at an angle θ_i, where $\tan\theta_i = (y_{i+1} - y_i)/d$. The resulting force on particle i thus has horizontal component $F_{\text{horiz}} = T\cos\theta_i$ and vertical component $F_{\text{vert}} = T\sin\theta_i$. Expanding in Taylor series and keeping only the terms linear in y_i and y_{i+1}, we obtain

$$F_{\text{horiz}} = T + O(y^2) \tag{42a}$$

$$F_{\text{vert}} = \frac{T}{d}(y_{i+1} - y_i) + O(y^3) \tag{42b}$$

where $O(y^2)$ ad $O(y^3)$ indicate the order of the neglected terms. (You should make sure that you understand the reasoning here.) Analogous reasoning gives the force on particle i from the element of string connecting it to particle $i-1$:

$$F_{\text{horiz}} = -T + O(y^2) \tag{43a}$$

$$F_{\text{vert}} = \frac{T}{d}(y_{i-1} - y_i) + O(y^3) \tag{43b}$$

The net horizontal force on particle i is therefore zero (in this linear approximation), while the net vertical force on particle i is

$$F_i = \frac{T}{d}(y_{i-1} - 2y_i + y_{i+1}) \tag{44}$$

The resulting equations are therefore *identical* to the equations (31) found in the previous section for the *longitudinal* oscillations of masses connected by springs, with the spring constant k replaced by the quantity T/d.

So we need not repeat the solution; we already know it!

In Problem Set #3 you will consider yet another situation that gives rise to the same system of equations (in the linear approximation): namely, the *transverse* oscillations of masses connected by springs.

Next week we will look at the limit $n \to \infty$ of the problem treated in this section — namely, waves on a string of length L, with the mass of the string uniformly distributed along its length — and we will find standing-wave solutions corresponding to the functions

$$f_j(s) = \sin\left(\frac{\pi j s}{L}\right) \tag{45}$$

where s is now a *real* number satisfying $0 \le s \le L$, and j is now an *arbitrarily large* positive integer.