

HANDOUT #11: THE LAGRANGIAN APPROACH TO MECHANICS

These notes are intended to be read as a supplement to the handout from Gregory, *Classical Mechanics*, Chapter 12.

1 The basic set-up

Here I follow the notation of Gregory, Sections 12.1–12.3, with a few additions.

The basic set-up is as follows: We have N particles moving in \mathbb{R}^3 , with position coordinates $\mathbf{r}_1, \dots, \mathbf{r}_N$. We assemble these coordinates into a single big vector $\mathbf{R} = (\mathbf{r}_1, \dots, \mathbf{r}_N) \in \mathbb{R}^{3N}$. Therefore, the **configuration space of the unconstrained system** is \mathbb{R}^{3N} .

If we wish, we can reinterpret this system as a *single* particle moving in the $3N$ -dimensional Euclidean space \mathbb{R}^{3N} ; the two interpretations are mathematically equivalent. (You might find this to be a strange way of thinking. But I personally find it easier to conceptualize a single particle moving in a high-dimensional space, rather than having to keep in my mind the simultaneous motion of many particles moving in 3-dimensional space. Perhaps this is a question of taste; but the reinterpretation as a single particle moving in a high-dimensional space clarifies many things, as we shall soon see.)

Now we impose k geometrical constraints:

$$\begin{aligned} f_1(\mathbf{R}, t) &= 0 \\ &\vdots \\ f_k(\mathbf{R}, t) &= 0 \end{aligned} \tag{1}$$

where f_1, f_2, \dots, f_k are specified functions. Here $0 \leq k < 3N$; we thus explicitly allow the case $k = 0$ (i.e. no constraints). Note also that we are now allowing *time-dependent* constraints. (Gregory does this starting in Section 12.6, but I find it convenient to do it from the beginning.) The **configuration space of the constrained system** at time t is thus

$$\mathcal{M}_t = \{ \mathbf{R} \in \mathbb{R}^{3N} : f_i(\mathbf{R}, t) = 0 \text{ for } i = 1, \dots, k \} . \tag{2}$$

If the constraints happen to be time-independent, then \mathcal{M}_t is the same set for all t , in which case we can refer to it simply as \mathcal{M} .

The constraints (1) are assumed to be *functionally independent* in the sense that the gradient vectors $\nabla f_1, \dots, \nabla f_k$ (where ∇ denotes partial differentiation with respect to \mathbf{R} , with t held fixed) are linearly independent at each point of the manifold \mathcal{M}_t . Under this assumption, the configuration space \mathcal{M}_t is a smooth submanifold of \mathbb{R}^{3N} , of dimension $n = 3N - k$; its normal vectors at any point are given precisely by the gradient vectors $\nabla f_1, \dots, \nabla f_k$.

If we wish, we can reinterpret this system as a single particle moving in the $3N$ -dimensional Euclidean space \mathbb{R}^{3N} subject to the constraints (1): that is, a single particle constrained to move on the (possibly moving) manifold $\mathcal{M}_t \subseteq \mathbb{R}^{3N}$. So, from this point of view, what we have is completely analogous to the elementary cases of a particle constrained to move on a (possibly moving) wire or surface in \mathbb{R}^3 ; the only difference is that here the ambient space has dimension $3N$ rather than just 3, and the manifold \mathcal{M}_t has dimension $3N - k$ rather than just 1 or 2.

We now parametrize the manifold(s) \mathcal{M}_t by **generalized coordinates** q_1, \dots, q_n , which we assemble into a big vector $\mathbf{q} = (q_1, \dots, q_n)$. “Parametrizing the manifold” means that we have written a function

$$\mathbf{R} = \mathbf{R}(\mathbf{q}, t) \tag{3}$$

such that $\mathbf{q} \mapsto \mathbf{R}(\mathbf{q}, t)$ maps \mathbf{q} -space smoothly and bijectively onto \mathcal{M}_t (for each t). Please note that this parametrization is allowed to be time-dependent, even if the constraints are time-independent!¹ We can also write the parametrization (3) as

$$\mathbf{r}_i = \mathbf{r}_i(\mathbf{q}, t) \quad \text{for } i = 1, \dots, N, \tag{4}$$

in which we make explicit the individual particles composing the system,

I stress that we are free to use *any parametrization we want*: any generalized coordinates q_1, \dots, q_n that correctly parametrize the manifold \mathcal{M}_t are permitted. This freedom to use *arbitrary* generalized coordinates is a notable feature of the Lagrangian approach to mechanics. (It is in a certain sense a precursor to the general coordinate invariance in Einstein’s general relativity.)

As explained in class, this parametrization of the manifold \mathcal{M}_t is intended to be *local*. There may or may not exist a single coordinate system that parametrizes the whole manifold \mathcal{M}_t .² Rather, we divide the manifold \mathcal{M}_t into *coordinate patches* (nonempty open sets), and each patch in \mathcal{M}_t is mapped smoothly and bijectively onto a nonempty open set in \mathbf{q} -space (that is, in \mathbb{R}^n). Since the equations of motion of classical mechanics are *differential equations*, this local approach suffices for our purposes: the system remains within one coordinate patch for some interval of time, and that allows us to perform any differentiations with respect to time and/or space that we may require.

¹This freedom is useful even if there are no constraints: for instance, it allows us to use a *rotating coordinate system* if we wish. (Rotating coordinate systems will be discussed later in this course, and are important in the study of rigid-body motion.)

²**Example:** It is *not* possible to parametrize the unit sphere in \mathbb{R}^3 with a single coordinate system. (It takes some topological work to prove this, but it is true.) For instance, spherical coordinates (θ, φ) are singular at the north and south poles, since the azimuthal (“longitude”) angle φ is ambiguous there. Or one could use the stereographic projection: this gives a good parametrization of the whole sphere *except the south pole*.

Our goal is now to obtain the equations of motion for the generalized coordinates \mathbf{q} . To do this, we start from the Newtonian equations of motion, and then do two things: First, we eliminate all reference to the constraint forces; and second, we eliminate reference to \mathbf{R} in favor of \mathbf{q} .

2 D'Alembert's principle of virtual work

This explanation is intended as a simplification of Gregory, Section 12.4.

The Newtonian equations of motion for our system are

$$m_i \ddot{\mathbf{r}}_i = \mathbf{F}_i^S + \mathbf{F}_i^C \quad \text{for } i = 1, \dots, N \quad (5)$$

where \mathbf{F}_i^S is the **specified force** acting on particle i , and \mathbf{F}_i^C is the **constraint force** acting on particle i .

The principle we use is simple: *the constraint force always acts in a direction perpendicular to the manifold \mathcal{M}_t* . Therefore, if $\mathbf{B} = (\mathbf{b}_1, \dots, \mathbf{b}_N)$ is a tangent vector to the manifold \mathcal{M}_t , then the constraint force $\mathbf{F}^C = (\mathbf{F}_1^C, \dots, \mathbf{F}_i^C)$ satisfies

$$\mathbf{F}^C \cdot \mathbf{B} = 0, \quad (6)$$

or equivalently (writing it out in terms of the individual particles)

$$\sum_{i=1}^N \mathbf{F}_i^C \cdot \mathbf{b}_i = 0. \quad (7)$$

This is the essential content of **d'Alembert's principle of virtual work**.

[More precisely, what we should have said is: If $\mathbf{B} = (\mathbf{b}_1, \dots, \mathbf{b}_N)$ is a tangent vector to the manifold \mathcal{M}_t at some specified point $\mathbf{R} \in \mathcal{M}_t$, then if the system is located at \mathbf{R} at time t , then the constraint force $\mathbf{F}^C = (\mathbf{F}_1^C, \dots, \mathbf{F}_i^C)$ satisfies $\mathbf{F}^C \cdot \mathbf{B} = 0$. That is what it means to say that “the constraint force always acts in a direction perpendicular to the manifold \mathcal{M}_t ”.]

We are very familiar with this principle in the elementary cases of a particle constrained to move frictionlessly on a (possibly moving) wire or surface in \mathbb{R}^3 ; indeed, the statement that the the constraint force always acts in a direction perpendicular to the wire or surface is merely the *definition* of “frictionless”! D'Alembert's principle of virtual work is simply the generalization of this principle to an arbitrary system of particles subjected to an arbitrary collection (1) of geometrical constraints.

Indeed, if we reinterpret our system as a single particle constrained to move on the (possibly moving) manifold $\mathcal{M}_t \subseteq \mathbb{R}^{3N}$, then d'Alembert's principle of virtual work is *precisely* the statement that the the constraint force always acts in a direction perpendicular to the constraint manifold.

Remark. It seems to me that the usual physicists’ way of explaining d’Alembert’s principle of virtual work — referring to “kinematically possible velocities \mathbf{v}_i^* ” or “kinematically possible infinitesimal displacements $d\mathbf{r}_i$ ” as Gregory does in Section 12.4 — just makes things unnecessarily confusing. And this approach becomes even more confusing when one takes up time-dependent constraints, as Gregory does in Section 12.6: then he has to clarify that the “kinematically possible velocities \mathbf{v}_i^* ” are not actually possible velocities of the system *with the actual moving constraints*, but are merely possible velocities of the system *with the constraints fixed to be those of one fixed time t* ; and he ends with the “mysterious statement” that “moving constraints do real work but no virtual work”.

All this is indeed unnecessarily mysterious! The point is simply that *the constraint force always acts in a direction perpendicular to the manifold \mathcal{M}_t* : this is true for *all* (frictionless) constraints, whether or not they are time-dependent. D’Alembert’s principle holds whether or not the constraints are time-dependent.

Now, if the constraints happen to be time-independent, then the velocities $\dot{\mathbf{R}}$ are tangent to the manifold \mathcal{M} , so the constraint force (which is perpendicular to the manifold \mathcal{M}) does no work: $\mathbf{F}^C \cdot \dot{\mathbf{R}} = 0$. On the other hand, if the constraints are time-dependent, then the velocities $\dot{\mathbf{R}}$ need not be tangent to the manifold \mathcal{M}_t , and the constraint force *can* do work. End of story.

So let $\mathbf{B} = (\mathbf{b}_1, \dots, \mathbf{b}_N)$ be any tangent vector to the manifold \mathcal{M}_t , and let us take the dot product of the Newtonian equations of motion (5) with the vector \mathbf{B} . That is, let us take the dot product of the i th equation in (5) with the vector \mathbf{b}_i , and then sum over i . Because of d’Alembert’s principle (7), the constraint forces \mathbf{F}_i^C drop out, and we obtain

$$\sum_{i=1}^N m_i \ddot{\mathbf{r}}_i \cdot \mathbf{b}_i = \sum_{i=1}^N \mathbf{F}_i^S \cdot \mathbf{b}_i . \quad (8)$$

We have now completed our first task: eliminating reference to the constraint forces.

Note that we have an equation of the type (8) for *each* tangent vector \mathbf{B} . Since the manifold \mathcal{M}_t is n -dimensional, there are n linearly independent tangent vectors at each point, so we actually have n independent equations of motion. This is exactly what we need for a system with n degrees of freedom!

3 Lagrange’s equations

We now turn to our second task: eliminating reference to \mathbf{R} in favor of \mathbf{q} .

First we need to find some tangent vectors $\mathbf{B} = (\mathbf{b}_1, \dots, \mathbf{b}_N)$ to the manifold \mathcal{M}_t ; in particular, we would like to find n linearly independent tangent vectors. But this is easy: for any index j ($j = 1, \dots, n$), the vector $\partial\mathbf{R}/\partial q_j$ [cf. (3)] is a tangent vector to the manifold \mathcal{M}_t — namely, it is the tangent vector in the direction of increasing coordinate q_j (with all other coordinates held fixed). So in our equations of motion (8) we can substitute $\mathbf{B} = \partial\mathbf{R}/\partial q_j$, or in other words $\mathbf{b}_i = \partial\mathbf{r}_i/\partial q_j$: we get

$$\sum_{i=1}^N m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial\mathbf{r}_i}{\partial q_j} = \sum_{i=1}^N \mathbf{F}_i^S \cdot \frac{\partial\mathbf{r}_i}{\partial q_j} \quad \text{for } j = 1, \dots, n . \quad (9)$$

The right-hand side of this equation is called the **generalized force** Q_j corresponding to the coordinate q_j : that is, we define

$$Q_j \stackrel{\text{def}}{=} \sum_{i=1}^N \mathbf{F}_i^S \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} . \quad (10)$$

With this definition, our equations of motion have become

$$\sum_{i=1}^N m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = Q_j \quad \text{for } j = 1, \dots, n . \quad (11)$$

We now need to work a bit on the left-hand side of this equation.

Before proceeding further, let me clarify some important conceptual and notational issues, which if not properly understood can lead to serious confusion.

We are going to be considering functions $f(\mathbf{q}, \dot{\mathbf{q}}, t)$ and writing things like

$$\frac{\partial f}{\partial q_j}, \quad \frac{\partial f}{\partial \dot{q}_j}, \quad \frac{\partial f}{\partial t} \quad (12)$$

and

$$\frac{d}{dt} f(\mathbf{q}, \dot{\mathbf{q}}, t) \quad \text{or simply } \frac{df}{dt} \quad \text{or } \dot{f} . \quad (13)$$

What do we *mean* by these expressions?

To explain what these expressions mean, let me take an example you already know well: consider a single particle moving in one dimension subject to a force law $F(x, v, t)$ that depends on the particle's position x , its velocity v , and time t . For example, the force law could be $F(x, v, t) = e^t x^2 v + (\cos t) x v^{17}$ or whatever. Then, by $\partial F / \partial x$ we simply mean the partial derivative of F with respect to x when v and t are held fixed. Likewise, by $\partial F / \partial v$ we simply mean the partial derivative of F with respect to v when x and t are held fixed. And by $\partial F / \partial t$ we mean the partial derivative of F with respect to t when x and v are held fixed. No ambiguity there: the quantities x , v and t are considered as *independent* variables in the function $F(x, v, t)$, so we can carry out partial differentiation as usual.

Next we imagine that the particle is moving according to some specified function $x(t)$, so that its velocity is $v(t) = \dot{x}(t)$; and *we insert these into the force law*, that is, we consider the *composite function* $F(x(t), \dot{x}(t), t)$, which is now a function of the single variable t . Whenever we write dF/dt (or \dot{F}), we implicitly mean the time derivative of such a composite function. It can of course be computed from the chain rule:

$$\frac{dF}{dt} = \frac{\partial F}{\partial x} \dot{x} + \frac{\partial F}{\partial v} \ddot{x} + \frac{\partial F}{\partial t} \quad (14)$$

where \dot{x} and \ddot{x} are the first and second time derivatives of the specified function $x(t)$. We sometimes call (14) the **total derivative** of F with respect to t ; but all it means is the time derivative of the composite function $F(x(t), \dot{x}(t), t)$, where the underlying motion $x(t)$ is *implicitly understood*.

The interpretation of partial derivatives and total derivatives in our Lagrangian formalism will be *exactly the same*. Namely, we consider a quantity $f(\mathbf{q}, \dot{\mathbf{q}}, t)$ that

depends on the generalized coordinates \mathbf{q} , the generalized velocities $\dot{\mathbf{q}}$, and time t — where \mathbf{q} , $\dot{\mathbf{q}}$ and t are here considered as *independent* variables, just as x , v and t were considered as independent variables in the expression $F(x, v, t)$. [Don't let yourself be confused by the notational quirk that we write the generalized velocity with a dot (i.e. $\dot{\mathbf{q}}$) rather than with a new letter (e.g. \mathbf{w}). It is still considered as an independent variable in the function $f(\mathbf{q}, \dot{\mathbf{q}}, t)$, exactly as v was in the function $F(x, v, t)$.] Then the partial derivatives (12) have their standard meanings: $\partial f/\partial q_j$ means the partial derivative with respect to q_j when $q_1, \dots, q_{j-1}, q_{j+1}, \dots, q_n$, $\dot{\mathbf{q}}$ and t are all held fixed; $\partial f/\partial \dot{q}_j$ means the partial derivative with respect to \dot{q}_j when \mathbf{q} , $\dot{q}_1, \dots, \dot{q}_{j-1}, \dot{q}_{j+1}, \dots, \dot{q}_n$ and t are all held fixed; and $\partial f/\partial t$ means the partial derivative with respect to t when \mathbf{q} and $\dot{\mathbf{q}}$ are all held fixed. This explains the meaning of the partial derivatives (12).

The total derivative (13) is likewise interpreted exactly as it was in our elementary example. That is, we imagine that the system is moving according to some specified function $\mathbf{q}(t)$, so that its generalized velocity is $\dot{\mathbf{q}}(t) = \frac{d}{dt}\mathbf{q}(t)$; and we insert these into the function f , so that we consider the composite function $f(\mathbf{q}(t), \dot{\mathbf{q}}(t), t)$, which is now a function of the single variable t . Whenever we write df/dt (or \dot{f}), we implicitly mean the time derivative of such a composite function. It can of course be computed from the chain rule:

$$\frac{df}{dt} = \sum_{j=1}^n \frac{\partial f}{\partial q_j} \dot{q}_j + \sum_{j=1}^n \frac{\partial f}{\partial \dot{q}_j} \ddot{q}_j + \frac{\partial f}{\partial t}. \quad (15)$$

We again call this the **total derivative** of f with respect to t ; but all it means is the time derivative of the composite function $f(\mathbf{q}(t), \dot{\mathbf{q}}(t), t)$, where the underlying motion $\mathbf{q}(t)$ is implicitly understood.

Another way of saying this is that the total-derivative operator d/dt , when it is acting on a function $f(\mathbf{q}, \dot{\mathbf{q}}, t)$, is a shorthand for the partial-differential operator

$$\frac{d}{dt} \stackrel{\text{def}}{=} \sum_{j=1}^n \dot{q}_j \frac{\partial}{\partial q_j} + \sum_{j=1}^n \ddot{q}_j \frac{\partial}{\partial \dot{q}_j} + \frac{\partial}{\partial t}. \quad (16)$$

The second and last step in our analysis is to transform the left-hand side of (11) to eliminate reference to \mathbf{R} and its time derivatives in favor of \mathbf{q} and its time derivatives. In 1788 Lagrange discovered how to do this; his formula expresses the left-hand side of (11) in terms of the **kinetic energy** of the system, as follows:

Proposition (Lagrange). The left-hand side of (11) can be written as

$$\sum_{i=1}^N m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j}, \quad (17)$$

where $T(\mathbf{q}, \dot{\mathbf{q}}, t)$ is the kinetic energy

$$T = \sum_{i=1}^N \frac{1}{2} m_i \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i \quad (18)$$

rewritten as a function of \mathbf{q} , $\dot{\mathbf{q}}$ and t .

Before proving this Proposition, let us compute some examples to illustrate how it works:

Example 1. *A single particle in Cartesian coordinates.* The generalized coordinates are $\mathbf{q} = (x, y, z)$ and the generalized velocities are $\dot{\mathbf{q}} = (\dot{x}, \dot{y}, \dot{z})$. The kinetic energy is

$$T(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2). \quad (19)$$

We have

$$\frac{\partial T}{\partial \dot{x}} = m\dot{x} \quad (20)$$

and hence

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}} \right) = m\ddot{x}. \quad (21)$$

On the other hand, we have

$$\frac{\partial T}{\partial x} = 0. \quad (22)$$

Hence

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}} \right) - \frac{\partial T}{\partial x} = m\ddot{x}, \quad (23)$$

as it should be. (And likewise for y and z .) \square

Example 2. *A single particle in plane polar coordinates.* The generalized coordinates are $\mathbf{q} = (r, \varphi)$ and the generalized velocities are $\dot{\mathbf{q}} = (\dot{r}, \dot{\varphi})$. The particle's velocity vector is $\mathbf{v} = \dot{r}\hat{e}_r + r\dot{\varphi}\hat{e}_\varphi$ [where \hat{e}_r and \hat{e}_φ are the unit vectors in the radial and tangential directions, respectively], so that the particle's kinetic energy $T = \frac{1}{2}m\mathbf{v} \cdot \mathbf{v}$ is

$$T(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2). \quad (24)$$

Let us now carry out the differentiations requested by Lagrange:

Radial coordinate. We have

$$\frac{\partial T}{\partial \dot{r}} = m\dot{r} \quad (25)$$

and hence

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{r}} \right) = m\ddot{r}. \quad (26)$$

On the other hand, we have

$$\frac{\partial T}{\partial r} = mr\dot{\varphi}^2. \quad (27)$$

Hence

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{r}} \right) - \frac{\partial T}{\partial r} = m(\ddot{r} - r\dot{\varphi}^2), \quad (28)$$

i.e. exactly m times the radial component of the acceleration. (Which is correct, because $\partial \mathbf{r} / \partial r = \hat{e}_r$, so that the generalized force in the radial direction is indeed the radial component of the force.)

Tangential coordinate. We have

$$\frac{\partial T}{\partial \dot{\varphi}} = mr^2 \dot{\varphi} \quad (29)$$

[note, by the way, that this equals the angular momentum] and hence

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\varphi}} \right) = m(r^2 \ddot{\varphi} + 2r\dot{r}\dot{\varphi}), \quad (30)$$

while

$$\frac{\partial T}{\partial \varphi} = 0. \quad (31)$$

Hence

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\varphi}} \right) - \frac{\partial T}{\partial \varphi} = mr(r\ddot{\varphi} + 2\dot{r}\dot{\varphi}), \quad (32)$$

i.e. exactly mr times the tangential component of the acceleration. (Which is again correct, because $\partial \mathbf{r} / \partial \varphi = r \hat{e}_\varphi$, so that the generalized force in the tangential direction is r times the tangential component of the force.)

In summary: To apply the Lagrangian method in some strange coordinate system, one simply needs to find the formula for the *kinetic energy* in those coordinates; that is, one basically needs to find the formula for (the squared magnitude of) the *velocity* in those coordinates. Then the ‘‘Lagrangian machine’’ gives you, in essence, the formula for acceleration in that coordinate system, without any further thought. \square

Let us now start the proof of Lagrange’s formula (17). Clearly we first need to understand what $T(\mathbf{q}, \dot{\mathbf{q}}, t)$ is; then we need to compute its derivatives with respect to \mathbf{q} and $\dot{\mathbf{q}}$. And to understand the kinetic energy T defined by (18), we first need to understand the particle velocities $\dot{\mathbf{r}}_i$. To do this, let us begin by recalling that we have parametrized the manifold \mathcal{M}_t with functions $\mathbf{r}_i(\mathbf{q}, t)$ as in (3)/(4). We can use this to compute the particle velocities:

Step 1. *Particle velocities.* If the system moves according to some specified function $\mathbf{q}(t)$, then the particle i has a position $\mathbf{r}_i(\mathbf{q}(t), t)$ and hence a velocity $\dot{\mathbf{r}}_i$ [note that this is a total derivative in the sense just explained!] that can be computed from the chain rule:

$$\dot{\mathbf{r}}_i = \sum_{j=1}^n \frac{\partial \mathbf{r}_i}{\partial q_j} \dot{q}_j + \frac{\partial \mathbf{r}_i}{\partial t} \quad (33)$$

where the partial derivatives have the meaning just explained. [This corresponds to Gregory’s equation (12.7) but has an extra term $\partial \mathbf{r}_i / \partial t$ because we are allowing time-dependent constraints and time-dependent parametrizations; this extra term is included in Gregory’s equation (12.21).]

Step 2. *Partial-differentiating the particle velocities.* We now forget temporarily about the specified motion $\mathbf{q}(t)$, and we simply consider (33) as defining a function $\dot{\mathbf{r}}_i$ of the variables

\mathbf{q} , $\dot{\mathbf{q}}$ and t , all considered as independent. We can understand these dependencies by looking at the right-hand side of (33): since \mathbf{r}_i is a function of \mathbf{q} and t but not of $\dot{\mathbf{q}}$ [cf. (4)], we see that $\partial\mathbf{r}_i/\partial q_j$ and $\partial\mathbf{r}_i/\partial t$ are likewise functions of \mathbf{q} and t but not of $\dot{\mathbf{q}}$. The only place where dependence on $\dot{\mathbf{q}}$ occurs is in the explicit appearance of \dot{q}_j on the right-hand side of (33). We therefore see that the particle velocities $\dot{\mathbf{r}}_i$ are *linear*³ in the generalized velocities \dot{q}_j , and we have the curious-looking formula

$$\frac{\partial\dot{\mathbf{r}}_i}{\partial\dot{q}_j} = \frac{\partial\mathbf{r}_i}{\partial q_j}. \quad (34)$$

You may, if you wish, choose to *remember* this formula by the mnemonic “cancel the dots”; but I want to stress that “cancelling the dots” is *not* in general a valid procedure, and that the formula (34) is *not* at all trivial.⁴

Step 3. An identity for $\frac{d}{dt}\left(\frac{\partial\mathbf{r}_i}{\partial q_j}\right)$. For later use, let us compute the total time derivative of $\partial\mathbf{r}_i/\partial q_j$. Since $\partial\mathbf{r}_i/\partial q_j$ is a function of \mathbf{q} and t but not of $\dot{\mathbf{q}}$, we have by the chain rule

$$\frac{d}{dt}\left(\frac{\partial\mathbf{r}_i}{\partial q_j}\right) = \sum_{k=1}^n \frac{\partial^2\mathbf{r}_i}{\partial q_j \partial q_k} \dot{q}_k + \frac{\partial^2\mathbf{r}_i}{\partial q_j \partial t} \quad (35a)$$

$$= \frac{\partial}{\partial q_j} \left[\sum_{k=1}^n \frac{\partial\mathbf{r}_i}{\partial q_k} \dot{q}_k + \frac{\partial\mathbf{r}_i}{\partial t} \right] \quad (35b)$$

$$= \frac{\partial\dot{\mathbf{r}}_i}{\partial q_j} \quad (35c)$$

where the last equality used (33) but with the dummy summation variable renamed from j to k .

Step 4. *Partial-differentiating one term in the kinetic energy with respect to \dot{q}_j .* The kinetic energy is a sum of terms $\frac{1}{2}m_i\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i$; so let us consider one of these terms, and to lighten the notation let us drop the factor m_i ; so let us consider $\frac{1}{2}\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i$. Lagrange tells us that we must partial-differentiate this with respect to \dot{q}_j :

$$\frac{\partial}{\partial\dot{q}_j} \left(\frac{1}{2}\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i \right) = \dot{\mathbf{r}}_i \cdot \frac{\partial\dot{\mathbf{r}}_i}{\partial\dot{q}_j} \quad (36a)$$

$$= \dot{\mathbf{r}}_i \cdot \frac{\partial\mathbf{r}_i}{\partial q_j} \quad (36b)$$

where the second equality used the identity (34).

³Or more precisely, *affine*.

⁴In particular, the Wikipedia article http://en.wikipedia.org/wiki/Lagrangian_mechanics#Kinetic_energy_relations is *wrong* to say that this formula holds simply “because q_j and \dot{q}_j are independent variables”; or at the very least, this is an *insufficient* explanation.

Step 5. *Taking the total time derivative of this.* Lagrange then tells us to take the total time derivative of the result (36):

$$\frac{d}{dt} \left[\frac{\partial}{\partial \dot{q}_j} \left(\frac{1}{2} \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i \right) \right] = \frac{d}{dt} \left[\dot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \right] \quad (37a)$$

$$= \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} + \dot{\mathbf{r}}_i \cdot \frac{d}{dt} \left(\frac{\partial \mathbf{r}_i}{\partial q_j} \right) \quad (37b)$$

$$= \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} + \dot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial q_j} \quad (37c)$$

where the last equality used the identity (35).

Step 6. *Partial-differentiating one term in the kinetic energy with respect to q_j .* Lagrange tells us that we must also partial-differentiate $\frac{1}{2} \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i$ with respect to q_j : clearly we have

$$\frac{\partial}{\partial q_j} \left(\frac{1}{2} \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i \right) = \dot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial q_j}. \quad (38)$$

Step 7. *Putting it all together.* Subtracting (38) from (37), we get

$$\frac{d}{dt} \left[\frac{\partial}{\partial \dot{q}_j} \left(\frac{1}{2} \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i \right) \right] - \frac{\partial}{\partial q_j} \left(\frac{1}{2} \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i \right) = \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}. \quad (39)$$

Multiplying this by m_i and summing over i , we obtain Lagrange's formula (17). \square

Why did this proof work? The key fact was identity (35):

$$\frac{d}{dt} \left(\frac{\partial \mathbf{r}_i}{\partial q_j} \right) = \frac{\partial \dot{\mathbf{r}}_i}{\partial q_j}. \quad (40)$$

That is, *in this case at least*, it is permissible to “commute the total derivative (d/dt or equivalently $\dot{}$) with the partial derivative $\partial/\partial q_j$ ”. Can we understand why this commutation of derivatives is valid?

Recall from (16) that the total-derivative operator d/dt , *when it is acting on a function $f(\mathbf{q}, \dot{\mathbf{q}}, t)$* , is simply a shorthand for the partial-differential operator

$$\frac{d}{dt} \stackrel{\text{def}}{=} \sum_{j=1}^n \dot{q}_j \frac{\partial}{\partial q_j} + \sum_{j=1}^n \ddot{q}_j \frac{\partial}{\partial \dot{q}_j} + \frac{\partial}{\partial t}. \quad (41)$$

And the operator on the right-hand side of (41) *does* commute with $\partial/\partial q_j$ (why?): that is, we have

$$\left[\sum_{j=1}^n \dot{q}_j \frac{\partial}{\partial q_j} + \sum_{j=1}^n \ddot{q}_j \frac{\partial}{\partial \dot{q}_j} + \frac{\partial}{\partial t} \right] \frac{\partial f}{\partial q_j} = \frac{\partial}{\partial q_j} \left[\sum_{j=1}^n \dot{q}_j \frac{\partial f}{\partial q_j} + \sum_{j=1}^n \ddot{q}_j \frac{\partial f}{\partial \dot{q}_j} + \frac{\partial f}{\partial t} \right] \quad (42)$$

for any function $f(\mathbf{q}, \dot{\mathbf{q}}, t)$. Moreover, if f is a function of \mathbf{q} , $\dot{\mathbf{q}}$ and t , then so is $\partial f/\partial q_j$; so (41) applies to it as well. We conclude that for *any* function $f(\mathbf{q}, \dot{\mathbf{q}}, t)$, we have

$$\frac{d}{dt} \left(\frac{\partial f}{\partial \dot{q}_j} \right) = \frac{\partial}{\partial q_j} \left(\frac{df}{dt} \right). \quad (43)$$

The identity (35)/(40) is just this general fact specialized to $f = \mathbf{r}_i$. (In this particular case f is in fact a function only of \mathbf{q} and t , not $\dot{\mathbf{q}}$.)

So this is the deep reason why Lagrange's formula (17) is valid.

In summary: Putting together the equation of motion (11) with Lagrange's identity (17), we have proven:

Lagrange's equations. In every motion of the constrained system compatible with Newton's equations (5), the coordinates $\mathbf{q}(t)$ must satisfy the system of equations

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} = Q_j \quad \text{for } j = 1, \dots, n, \quad (44)$$

where $T(\mathbf{q}, \dot{\mathbf{q}}, t)$ is the kinetic energy and

$$Q_j = \sum_{i=1}^N \mathbf{F}_i^S \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \quad (45)$$

is the generalized force.

Let us now further assume that the specified forces \mathbf{F}_i^S are **conservative**, i.e. that there exists a potential energy function $U(\mathbf{r}_1, \dots, \mathbf{r}_N, t)$ such that

$$\mathbf{F}_i^S = -\frac{\partial U}{\partial \mathbf{r}_i} \quad \text{for } i = 1, \dots, N. \quad (46)$$

By the notation $\partial U/\partial \mathbf{r}_i$ I mean the gradient of $U(\mathbf{r}_1, \dots, \mathbf{r}_N, t)$ with respect to the vector variable \mathbf{r}_i , with all the other particle positions (as well as t) being held fixed. This could also be written as $\nabla_i U$ or $\nabla_{\mathbf{r}_i} U$; I used the notation $\nabla_i U$ in Section 3.3.2 of Handout #6.

Note also that, in defining what we mean by a "conservative force", we ordinarily do *not* allow the potential energy to depend explicitly on t . (In Handout #6 I did *not* allow it.) But in the present context there is no harm in allowing this somewhat unusual generalization; Lagrange's equations (51) will hold regardless.

We now insert the parametrization (4) of the constraint manifold \mathcal{M}_t in terms of the generalized coordinates \mathbf{q} and time t , yielding

$$U(\mathbf{q}, t) \stackrel{\text{def}}{=} U(\mathbf{r}_1(\mathbf{q}, t), \dots, \mathbf{r}_N(\mathbf{q}, t)). \quad (47)$$

By the chain rule we have

$$\frac{\partial U}{\partial q_j} = \sum_{i=1}^N \frac{\partial U}{\partial \mathbf{r}_i} \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = - \sum_{i=1}^N \mathbf{F}_i^S \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = -Q_j. \quad (48)$$

So we can write the generalized force Q_j as a partial derivative of the potential energy with respect to the generalized coordinate q_j ,

$$Q_j = - \frac{\partial U}{\partial q_j}, \quad (49)$$

just as we write the ordinary force \mathbf{F}_i^S as a partial derivative of the potential energy with respect to the ordinary coordinate \mathbf{r}_i [cf. (46)].

It follows that if we define the **Lagrangian** $L(\mathbf{q}, \dot{\mathbf{q}}, t)$ as

$$L(\mathbf{q}, \dot{\mathbf{q}}, t) = T(\mathbf{q}, \dot{\mathbf{q}}, t) - U(\mathbf{q}, t) \quad (50)$$

[note the funny minus sign! — this is *not* the total energy $T + U$!], then we have:

Lagrange’s equations for a conservative system. In every motion of the constrained system compatible with Newton’s equations (5), the coordinates $\mathbf{q}(t)$ must satisfy the system of equations

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = 0 \quad \text{for } j = 1, \dots, n, \quad (51)$$

where the Lagrangian is defined as $L(\mathbf{q}, \dot{\mathbf{q}}, t) = T(\mathbf{q}, \dot{\mathbf{q}}, t) - U(\mathbf{q}, t)$, with $T(\mathbf{q}, \dot{\mathbf{q}}, t)$ being the kinetic energy and $U(\mathbf{q}, t)$ being the potential energy.

We can be even slightly more general (as Gregory points out in Section 12.7). Namely, suppose that there exists a function $U(\mathbf{q}, \dot{\mathbf{q}}, t)$ [note that it is now allowed to depend also on $\dot{\mathbf{q}}$!] such that the generalized force Q_j can be written as

$$Q_j = \frac{d}{dt} \left(\frac{\partial U}{\partial \dot{q}_j} \right) - \frac{\partial U}{\partial q_j}. \quad (52)$$

Then Lagrange’s equations (44) can clearly be written in the form (51) if we define

$$L(\mathbf{q}, \dot{\mathbf{q}}, t) = T(\mathbf{q}, \dot{\mathbf{q}}, t) - U(\mathbf{q}, \dot{\mathbf{q}}, t). \quad (53)$$

It no longer makes sense to call U a “potential energy”; but since it still plays the same *role* in the Lagrangian formalism as a potential energy, we call it a **velocity-dependent potential**. This generalization applies to one very important case, namely a charged particle moving in an electromagnetic field.

4 Symmetries and conservation laws

You should first read Gregory, Section 12.8 about energy conservation and Gregory, Section 12.9 about conservation of generalized momenta conjugate to cyclic coordinates. Then the explanation below is intended as a simplification of Gregory, Section 12.10.

Symmetry is a fundamental concept in mathematics; and over the last century or so it has come to play an increasingly central role in physics as well. We have already seen that **conservation laws** are extremely important in physics. We now come to a beautiful and surprising fact: the two themes of symmetries and conservation laws are intimately connected. Namely, to every (continuous) symmetry of a physical system (more precisely, a physical system expressible in Lagrangian form), there corresponds a conservation law. This connection was discovered by Emmy Noether in 1918, and is now known as **Noether's theorem**.⁵

A **symmetry** of a (physical or mathematical) object is a mapping that leaves the object invariant.⁶ For instance, the symmetries of the regular n -gon are rotations by multiples of $2\pi/n$ and reflections through lines passing through vertices of the n -gon and/or bisectors. Likewise, the symmetries of n -dimensional Euclidean space — that is, the mappings of \mathbb{R}^n to itself that preserve distances — are translations, rotations and reflections. Since the composition of two symmetries is obviously a symmetry, and the inverse of a symmetry is a symmetry, the symmetries of any given object obviously form a *group*, called the **symmetry group** of the object.⁷ The study of symmetries is thus intimately connected with **group theory**.

Here we are concerned with symmetries of a **dynamical law**. What we mean is this: Let \mathcal{K} be the set of all *kinematically possible motions* of a particular physical system, and let \mathcal{D} be the subset of \mathcal{K} consisting of all *dynamically allowed motions* (for the given dynamical law). For instance, for a system of N particles moving through \mathbb{R}^3 , the set \mathcal{K} consists of all smooth functions $\mathbf{R}(t) = (\mathbf{r}_1(t), \dots, \mathbf{r}_N(t))$ from \mathbb{R} (time) into \mathbb{R}^{3N} (configuration space), while the set \mathcal{D} consists of only those motions that satisfy Newton's equations for the given force law. Then any mapping $\mathcal{M}: \mathcal{K} \rightarrow \mathcal{K}$ for which $\mathcal{M}[\mathcal{D}] \subseteq \mathcal{D}$ is called a **symmetry** of the given dynamical law.⁸

Example 1. Suppose we have a system of N particles moving through \mathbb{R}^3 , interacting through forces $\mathbf{F}_{i \leftarrow j}(\mathbf{r}_i, \mathbf{r}_j)$ that depend only on $\mathbf{r}_i - \mathbf{r}_j$. (We assume that there are *no* external forces.) Then the dynamical law is invariant under **spatial translation** (or more precisely, simultaneous spatial translation of all the particles) — that is, under the mapping

⁵Emmy Noether (1882–1935) was an important German mathematician of the early twentieth century. Though her main contributions were to abstract algebra (**Noetherian rings** are named after her), her 1918 foray into theoretical physics was also extraordinarily important for the subsequent development of physics.

⁶For this reason, the word **invariance** is also used as a synonym of “symmetry”.

⁷Thus, the symmetries of the regular n -gon form the *dihedral group* D_{2n} ; and the symmetries of n -dimensional Euclidean space form the *Euclidean group* $E(n)$.

⁸Usually we will also require that the mapping \mathcal{M} be invertible, i.e. that it be a *bijection* of \mathcal{K} to itself. Then its restriction to \mathcal{D} will also be a bijection of \mathcal{D} to itself.

$\mathcal{M}: \mathcal{K} \rightarrow \mathcal{K}$ defined by

$$(\mathcal{M}\mathbf{R})(t) = \mathbf{R}(t) + (\mathbf{c}, \dots, \mathbf{c}) \quad (54)$$

where $\mathbf{c} \in \mathbb{R}^3$ is an arbitrary vector. We can also write the mapping \mathcal{M} as

$$\mathbf{r}_i \mapsto \mathbf{r}_i + \mathbf{c} \quad \text{for } i = 1, \dots, N. \quad (55)$$

The point is simply that if we simultaneously translate all the particles by the same displacement \mathbf{c} , then the relative positions $\mathbf{r}_i - \mathbf{r}_j$ are unchanged, hence the forces are unchanged; and since \mathbf{c} is a *constant* vector (i.e. independent of time), the accelerations $\ddot{\mathbf{r}}_i$ (which arise in Newton's laws) are also unchanged. Hence, if Newton's laws $m_i \ddot{\mathbf{r}}_i = \mathbf{F}_i$ we satisfied before the spatial translation, then they are also satisfied after.

Example 2. Suppose we have a system of N particles moving through \mathbb{R}^3 , interacting through forces $\mathbf{F}_{i \leftarrow j}(\mathbf{r}_i, \mathbf{r}_j)$ whose magnitude depends only on the distance $|\mathbf{r}_i - \mathbf{r}_j|$ and whose direction points along the line connecting the two particles:

$$\mathbf{F}_{i \leftarrow j}(\mathbf{r}_i, \mathbf{r}_j) = f_{i \leftarrow j}(|\mathbf{r}_i - \mathbf{r}_j|) \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|}. \quad (56)$$

(We again assume that there are *no* external forces.) Then the dynamical law is invariant under **spatial rotation**, namely under the mapping $\mathcal{M}: \mathcal{K} \rightarrow \mathcal{K}$ defined by

$$\mathbf{r}_i \mapsto M\mathbf{r}_i \quad \text{for } i = 1, \dots, N \quad (57)$$

where M is a 3×3 matrix representing a rotation of \mathbb{R}^3 .⁹ The reasoning is similar.

Example 3. Suppose we have a system of N particles moving through \mathbb{R}^3 , with forces \mathbf{F}_i that may now depend in a completely arbitrary way on the positions $\mathbf{r}_1, \dots, \mathbf{r}_N$ and the velocities $\dot{\mathbf{r}}_1, \dots, \dot{\mathbf{r}}_N$; the only thing we demand is that the forces *do not depend explicitly on the time t* . Then the dynamical law is invariant under **time translation**, namely under the mapping $\mathcal{M}: \mathcal{K} \rightarrow \mathcal{K}$ defined by

$$(\mathcal{M}\mathbf{R})(t) = \mathbf{R}(t + a) \quad (58)$$

where $a \in \mathbb{R}$ is arbitrary.

We now consider a physical system described by generalized coordinates $\mathbf{q} = (q_1, \dots, q_n)$ and obeying Lagrange's equations for some Lagrangian $L(\mathbf{q}, \dot{\mathbf{q}}, t)$. We shall develop Noether's theorem by starting from a special case and then gradually increasing the generality:

1) *Very special case.* Suppose that one of the coordinates, say q_s , is **cyclic**, i.e. that $\partial L / \partial q_s = 0$. This is equivalent to saying that L is *invariant under translation of the coordinate q_s* (all the other coordinates being held fixed): that is,

$$L(q_1, \dots, q_s, \dots, q_n, \dot{\mathbf{q}}, t) = L(q_1, \dots, q_s + c, \dots, q_n, \dot{\mathbf{q}}, t) \quad (59)$$

⁹Later in this term we will discuss such matrices: they belong to the so-called **special orthogonal group** $SO(3)$. Concretely, they are 3×3 matrices M satisfying $M^T M = M M^T = I$ (such matrices are called **orthogonal** matrices) and also $\det M = +1$. (The orthogonal matrices with $\det M = -1$ correspond to mirror reflections.)

for any arbitrary constant c . (Do you see why these two statements are *equivalent*, i.e. that each statement implies the other?)

On the other hand, we know (and it follows immediately from Lagrange's equations) that if $\partial L/\partial q_s = 0$, then

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_s} \right) = 0, \quad (60)$$

i.e. that the **conjugate momentum**

$$p_s \stackrel{\text{def}}{=} \frac{\partial L}{\partial \dot{q}_s} \quad (61)$$

is *conserved*.

So, in this case, a symmetry (invariance under translation of the coordinate q_s) implies a conservation law (conservation of the conjugate momentum p_s).

This is encouraging; let's try to generalize it.

2) *Nearly almost general case*. We consider some coordinate transformation $\mathbf{q}^* = \mathbf{q}^*(\mathbf{q}, t)$ that leaves the Lagrangian invariant, i.e.

$$L(\mathbf{q}^*, \dot{\mathbf{q}}^*, t) = L(\mathbf{q}, \dot{\mathbf{q}}, t). \quad (62)$$

[Important: L is here meant to have a fixed *functional form* on both sides of the equation. The assertion is that if we plug the arguments $\mathbf{q}^*, \dot{\mathbf{q}}^*, t$ into that function (where $\mathbf{q}^* \stackrel{\text{def}}{=} \mathbf{q}^*(\mathbf{q}, t)$), we get the same numerical value as when we plug the arguments $(\mathbf{q}, \dot{\mathbf{q}}, t)$ into that *same* function.]

Now let us further assume that there exists a whole *family* of such coordinate transformations, labeled by a continuous parameter ϵ and depending smoothly on ϵ , such that L is invariant under *every* individual transformation in this family. And we also assume that the identity transformation belongs to this family (for convenience let us label it as $\epsilon = 0$). Let us call these transformations $\mathbf{q}_\epsilon^* = \mathbf{q}_\epsilon^*(\mathbf{q}, t)$. We can then expand $\mathbf{q}_\epsilon^*(\mathbf{q}, t)$ in Taylor series in ϵ : using the fact that $\epsilon = 0$ corresponds to the identity transformation, we have

$$\mathbf{q}_\epsilon^*(\mathbf{q}, t) = \mathbf{q} + \epsilon \mathbf{f}(\mathbf{q}, t) + O(\epsilon^2) \quad (63)$$

for a suitable function $\mathbf{f}(\mathbf{q}, t)$ [namely, $\mathbf{f}(\mathbf{q}, t) = (\partial \mathbf{q}_\epsilon^* / \partial \epsilon)|_{\epsilon=0}$]; or writing explicitly the coordinates, we have

$$q_{j;\epsilon}^* = q_j + \epsilon f_j(\mathbf{q}, t) + O(\epsilon^2). \quad (64)$$

[The quantity \mathbf{f} is sometimes called the “infinitesimal transformation” corresponding to the given family of transformations.] We now insert \mathbf{q}_ϵ^* into the Lagrangian and expand in Taylor series in ϵ :

$$L(\mathbf{q}_\epsilon^*, \dot{\mathbf{q}}_\epsilon^*, t) = L(\mathbf{q}, \dot{\mathbf{q}}, t) + \epsilon \left[\sum_j \frac{\partial L}{\partial q_j} f_j + \sum_j \frac{\partial L}{\partial \dot{q}_j} \dot{f}_j \right] + O(\epsilon^2). \quad (65)$$

By hypothesis the Lagrangian is invariant under this transformation for all ϵ , so in particular the coefficient of ϵ^1 in the Taylor series (i.e. the square brackets in the preceding equation) must vanish:

$$\sum_j \left[\frac{\partial L}{\partial q_j} f_j + \frac{\partial L}{\partial \dot{q}_j} \dot{f}_j \right] = 0. \quad (66)$$

On the other hand, we can rewrite

$$\frac{\partial L}{\partial \dot{q}_j} \dot{f}_j = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} f_j \right) - f_j \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right); \quad (67)$$

and by Lagrange's equations we have

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) = \frac{\partial L}{\partial q_j}. \quad (68)$$

Plugging (67) and (68) into (66), we see that the first and last terms cancel, and we are left with simply

$$\sum_j \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} f_j \right) = 0. \quad (69)$$

In other words, the quantity

$$C = \sum_j \frac{\partial L}{\partial \dot{q}_j} f_j \quad (70)$$

is a constant of motion.

3) *Almost general case.* For Lagrange's *equations of motion* to be invariant under the coordinate transformation, it is certainly sufficient that L itself be invariant; but it is not necessary. In fact, if $L(\mathbf{q}, \dot{\mathbf{q}}, t)$ and $L'(\mathbf{q}, \dot{\mathbf{q}}, t)$ are two Lagrangians related by

$$L'(\mathbf{q}, \dot{\mathbf{q}}, t) = L(\mathbf{q}, \dot{\mathbf{q}}, t) + \frac{d}{dt} F(\mathbf{q}, t) \quad (71)$$

for some function $F(\mathbf{q}, t)$ of the generalized coordinates and time (but *not* of the generalized velocities), then L and L' lead to the *same* equations of motion; you will prove this in Problem 2(a) of Problem Set #6. So, in this situation, the two Lagrangians are "physically equivalent" even though $L' \neq L$.

So let us assume that we have a one-parameter family of transformations $\mathbf{q}_\epsilon^* = \mathbf{q}_\epsilon^*(\mathbf{q}, t)$ as in case #2, with infinitesimal transformation \mathbf{f} given by (63)/(64), under which the Lagrangian is invariant *up to a total time derivative*, i.e. that

$$L(\mathbf{q}_\epsilon^*, \dot{\mathbf{q}}_\epsilon^*, t) = L(\mathbf{q}, \dot{\mathbf{q}}, t) + \frac{d}{dt} F_\epsilon(\mathbf{q}, t) \quad (72)$$

for some functions $F_\epsilon(\mathbf{q}, t)$. Now we can expand $F_\epsilon(\mathbf{q}, t)$ in Taylor series in ϵ as

$$F_\epsilon(\mathbf{q}, t) = F_0(\mathbf{q}, t) + \epsilon \tilde{F}(\mathbf{q}, t) + O(\epsilon^2) \quad (73)$$

where $\tilde{F}(\mathbf{q}, t) = (\partial/\partial\epsilon) F_\epsilon(\mathbf{q}, t)|_{\epsilon=0}$; and since $\epsilon = 0$ is the identity transformation, we have $F_0(\mathbf{q}, t) = 0$. Inserting this into (72) we conclude that

$$L(\mathbf{q}_\epsilon^*, \dot{\mathbf{q}}_\epsilon^*, t) = L(\mathbf{q}, \dot{\mathbf{q}}, t) + \epsilon \frac{d}{dt} \tilde{F}(\mathbf{q}, t) + O(\epsilon^2). \quad (74)$$

Then, comparing (65) with (74), and following the same steps as in (66)–(69), we see that the quantity

$$C = \sum_j \frac{\partial L}{\partial \dot{q}_j} f_j - \tilde{F} \quad (75)$$

is a constant of motion, i.e. $dC/dt = 0$. (You should make sure to check the details yourself!) Case #2 of course corresponds to the special case when $\tilde{F} = 0$.

4) *General case (of sorts)*. Nowhere in the proof of case #2 above did we use the assumption that \mathbf{q}^* (and hence \mathbf{f}) is a function only of \mathbf{q} and t , rather than also of the velocities $\dot{\mathbf{q}}$. So we might as well dispense with that restriction. This more general transformation $\mathbf{q}^* = \mathbf{q}^*(\mathbf{q}, \dot{\mathbf{q}}, t)$ might better be termed a “path transformation” rather than a “coordinate transformation”. Our assumption is therefore that we have a one-parameter family of path transformations $\mathbf{q}_\epsilon^* = \mathbf{q}_\epsilon^*(\mathbf{q}, \dot{\mathbf{q}}, t)$, with infinitesimal transformation $\mathbf{f} = \mathbf{f}(\mathbf{q}, \dot{\mathbf{q}}, t)$ given by

$$\mathbf{q}_\epsilon^*(\mathbf{q}, \dot{\mathbf{q}}, t) = \mathbf{q} + \epsilon \mathbf{f}(\mathbf{q}, \dot{\mathbf{q}}, t) + O(\epsilon^2). \quad (76)$$

Likewise, nowhere in the proof of case #3 above did we use the assumption that F is independent of the velocities. So let us throw away that restriction too, and assume simply that

$$L(\mathbf{q}_\epsilon^*, \dot{\mathbf{q}}_\epsilon^*, t) = L(\mathbf{q}, \dot{\mathbf{q}}, t) + \epsilon \frac{d}{dt} \tilde{F}(\mathbf{q}, \dot{\mathbf{q}}, t) + O(\epsilon^2) \quad (77)$$

for some function $\tilde{F}(\mathbf{q}, \dot{\mathbf{q}}, t)$. Once we allow \tilde{F} to be velocity-dependent, the connection with “physical equivalence” of Lagrangians is lost [see Problem 2(b) of Problem Set #6], so we shouldn’t really call this transformation a “symmetry” of the dynamical law — but once we’ve gotten this far, who cares? The key fact is that $C = \sum_j \frac{\partial L}{\partial \dot{q}_j} f_j - \tilde{F}$ is conserved nonetheless, under the hypothesis that the transformation (76) obeys (77).

All these results are special cases of **Noether’s theorem** (1918), which is of considerable importance not only in classical mechanics but in both classical and quantum field theory.

Example 1, revisited. Suppose that the Lagrangian is invariant under spatial translation in some fixed direction \mathbf{e} , namely

$$\mathbf{r}_i \mapsto \mathbf{r}_i + \epsilon \mathbf{e} \quad \text{for } i = 1, \dots, N. \quad (78)$$

Then the conserved quantity associated with this symmetry will be **linear momentum** (more precisely, its component in the direction \mathbf{e}).

To see this, consider the Lagrangian in Cartesian coordinates:

$$L(\mathbf{r}_1, \dots, \mathbf{r}_N, \dot{\mathbf{r}}_1, \dots, \dot{\mathbf{r}}_N, t) = \sum_{i=1}^N \frac{1}{2} m_i \dot{\mathbf{r}}_i^2 - U(\mathbf{r}_1, \dots, \mathbf{r}_N, t). \quad (79)$$

The kinetic energy $\sum_{i=1}^N \frac{1}{2} m_i \dot{\mathbf{r}}_i^2$ is manifestly invariant under spatial translation; so the Lagrangian is invariant under spatial translation if and only if the potential energy $U(\mathbf{r}_1, \dots, \mathbf{r}_N, t)$

is. Now, spatial translation (78) is of the form (63) considered in case #2 above, with $\mathbf{f}_i = \mathbf{e}$. The conserved quantity

$$C = \sum_j \frac{\partial L}{\partial \dot{q}_j} f_j \quad (80)$$

constructed by Noether's theorem is therefore

$$C = \sum_{i=1}^N \frac{\partial L}{\partial \dot{\mathbf{r}}_i} \cdot \mathbf{e} = \sum_{i=1}^N m_i \dot{\mathbf{r}}_i \cdot \mathbf{e} = \mathbf{P} \cdot \mathbf{e} \quad (81)$$

where \mathbf{P} is the total linear momentum of the system.

In the special case of a *single particle* in Cartesian coordinates, this example even falls into case #1 of Noether's theorem (i.e. a cyclic coordinate).

Example 2, revisited. Suppose that the Lagrangian is invariant under spatial rotation; then the conserved quantity associated with this symmetry will be **angular momentum**.

To see this, we again work with the Lagrangian in Cartesian coordinates. In Cartesian coordinates, rotation by angle ϵ about an axis \mathbf{e} (where \mathbf{e} is a unit vector) yields

$$\mathbf{r}_i^* = \mathbf{r}_i + \epsilon \mathbf{e} \times \mathbf{r}_i + O(\epsilon^2), \quad (82)$$

so that we are again in case #2, this time with

$$\mathbf{f}_i = \mathbf{e} \times \mathbf{r}_i. \quad (83)$$

As in Example 1 we have we

$$\frac{\partial L}{\partial \dot{\mathbf{r}}_i} = \frac{\partial T}{\partial \dot{\mathbf{r}}_i} = m_i \dot{\mathbf{r}}_i, \quad (84)$$

so the conserved quantity is

$$C = \sum_{i=1}^N \frac{\partial L}{\partial \dot{\mathbf{r}}_i} \cdot \mathbf{f}_i \quad (85a)$$

$$= \sum_{i=1}^N m_i \dot{\mathbf{r}}_i \cdot (\mathbf{e} \times \mathbf{r}_i) \quad (85b)$$

$$= \sum_{i=1}^N \mathbf{e} \cdot (\mathbf{r}_i \times m_i \dot{\mathbf{r}}_i) \quad (85c)$$

$$= \mathbf{e} \cdot \mathbf{L} \quad (85d)$$

where \mathbf{L} is the total angular momentum of the system. [Here we used the invariance of the scalar triple product under cyclic shifts of its three operands: $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$.]

Example 3, revisited. Suppose that the Lagrangian is invariant under time translation, i.e. that $\partial L / \partial t = 0$; then the conserved quantity associated with this symmetry will be **energy**.

The map \mathcal{M} is here defined by evolving the system forward ϵ in time, i.e. let

$$\mathbf{q}_\epsilon^*(t) = \mathbf{q}(t + \epsilon) = \mathbf{q}(t) + \epsilon \dot{\mathbf{q}}(t) + O(\epsilon^2), \quad (86)$$

so that the “infinitesimal transformation” \mathbf{f} is here

$$\mathbf{f} = \dot{\mathbf{q}}. \quad (87)$$

(This therefore needs the general case #4 of Noether’s theorem, in which we allow \mathbf{f} to depend on $\dot{\mathbf{q}}$ as well as \mathbf{q} and t .) Then

$$L(\mathbf{q}_\epsilon^*, \dot{\mathbf{q}}_\epsilon^*, t) = L(\mathbf{q}, \dot{\mathbf{q}}, t) + \epsilon \left[\sum_j \frac{\partial L}{\partial q_j} \dot{q}_j + \sum_j \frac{\partial L}{\partial \dot{q}_j} \ddot{q}_j \right] + O(\epsilon^2). \quad (88)$$

Please note now that the quantity in square brackets equals dL/dt *provided that* $\partial L/\partial t = 0$. Therefore, if $\partial L/\partial t = 0$ we conclude that case #4 of Noether’s theorem, with $\tilde{F} = L$. It follows the conserved quantity is

$$C = \sum_j \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j - L \quad (89)$$

— which is precisely the “energy function h ” whose conservation (when $\partial L/\partial t = 0$) was already discussed in Gregory, Section 12.8. So we now see that what really underlies the conservation of h is invariance under time translation.

You will see some further applications of Noether’s theorem in Problems 4 and 5 of Problem Set #6.