MATHEMATICS 0054 (Analytical Dynamics)
YEAR 2019–2020, TERM 2 (SPRING 2020 EXAM)

MODEL SOLUTIONS TO FINAL EXAM

1. A chain of linear density (mass per unit length) σ lies in a heap on the floor. You grab an end and pull horizontally with a constant force F, starting at time 0. Assume that the chain is greased, so that it has no friction with itself.

   (a) What is the position x of the end of the chain, as a function of time t, while it is unravelling?

   (b) What is the kinetic energy of the chain at time t?

   (c) Between time 0 and time t, how much work have you done?

   (d) Compare your answers to (b) and (c). If they are the same, explain why; and if one of them is larger than the other, also explain why.

UNSEEN: (a) 13 points, (b) 4 points, (c) 4 points, (d) 4 points.

SOLUTION:

(a) Apply the force-momentum theorem to the chain. At time t, the moving part of the chain has length x and hence mass σx, and it has velocity $\dot{x}$. So the total momentum of the system at time t is $p = \sigma x \dot{x}$. On the other hand, the total external force on the system is F. So the force-momentum theorem gives

$$\frac{d}{dt}(\sigma x \dot{x}) \;=\; F \;.$$

Integrating with respect to time and using the initial condition $x(0) = 0$, we find

$$\sigma x \dot{x} \;=\; Ft \;.$$

This is a separable differential equation:

$$\sigma x \, dx \;=\; Ft \, dt$$

and hence (using again the initial condition $x(0) = 0$)

$$\frac{1}{2}\sigma x^2 = \frac{1}{2}Ft^2 ,$$

or in other words

$$x(t) = \sqrt{F/\sigma}\, t .$$

So the end of the chain moves at constant velocity $\sqrt{F/\sigma}$.

(b) The kinetic energy is

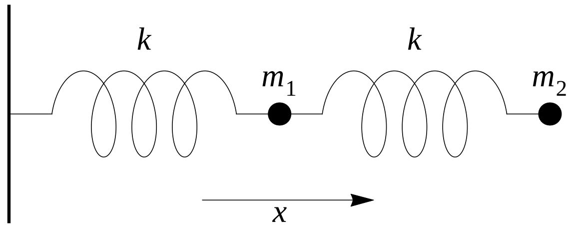
$$\begin{aligned} K(t) &= \frac{1}{2}\,(\sigma x)\,\dot{x}^2 \\ &= \frac{1}{2}\,F^{3/2}\sigma^{-1/2}\,t . \end{aligned}$$

(c) The force is a constant F, so the work done is the force times the displacement, i.e.

$$\begin{aligned} \text{work} &= Fx \\ &= F^{3/2}\sigma^{-1/2}\,t . \end{aligned}$$

(d) So the work done is *twice* the final kinetic energy. The remaining energy was lost in inelastic collions between successive links of the chain as they were put into motion.

2. Two particles, of masses m_1 and m_2, respectively, are connected to a fixed wall by springs of spring constant k, as shown in the diagram:



The particles move horizontally. Let x_1 and x_2 be the displacements of the two particles from their equilibrium positions.

(a) Derive the equations of motion. (You may use either Newtonian or Lagrangian methods.)

(b) Find the frequencies of the normal modes.

(c) In the special case $m_1/m_2 = 3/2$, find the eigenvectors corresponding to the normal modes.

SIMILAR SEEN: (a) 5 points, (b) 10 points, (c) 10 points

SOLUTION:

(a) By Newtonian methods we easily get

$$\begin{aligned} m_1\ddot{x}_1 &= -kx_1 + k(x_2 - x_1) \\ m_2\ddot{x}_2 &= -k(x_2 - x_1) \end{aligned}$$

or in other words

$$M\ddot{\mathbf{x}} + K\mathbf{x} = 0$$

where

$$\begin{aligned} \mathbf{x} &= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ M &= \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \\ K &= \begin{pmatrix} 2k & -k \\ -k & k \end{pmatrix} \end{aligned}$$

(b) Inserting the Ansatz $\mathbf{x}(t) = \mathbf{e}\, e^{i\omega t}$, we obtain the generalized eigenvalue problem

$$(K - \omega^2 M)\mathbf{e} \;=\; 0\ .$$

So the frequencies of the normal modes are $\omega = \sqrt{\lambda}$ where λ are the roots of the characteristic equation $\det(K - \lambda M) = 0$. We have

$$\det(K-\lambda M) \;=\; \det\begin{pmatrix} 2k - \lambda m_1 & -k \\ -k & k - \lambda m_2 \end{pmatrix} \;=\; m_1 m_2 \lambda^2 - (m_1+2m_2)k\lambda + k^2\ ,$$

so by the quadratic formula the roots are

$$\lambda_\pm \;=\; \frac{m_1 + 2m_2 \pm \sqrt{m_1^2 + 4m_2^2}}{2m_1 m_2}\, k$$

and the frequencies of the normal modes are $\omega_\pm = \sqrt{\lambda_\pm}$.

(c) Let us write $m_1 = 3m$, $m_2 = 2m$, so that

$$\lambda_+ \;=\; \frac{k}{m}$$

$$\lambda_- \;=\; \frac{k}{6m}$$

Then the eigenvector $\mathbf{e}_+$ satisfies $(K - \lambda_+ M)\mathbf{e}_+ = 0$, i.e.

$$\begin{pmatrix} -k & -k \\ -k & -k \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \;=\; 0\ ,$$

hence $\mathbf{e}_+$ is proportional to $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. And the eigenvector $\mathbf{e}_-$ satisfies $(K - \lambda_- M)\mathbf{e}_- = 0$, i.e.

$$\begin{pmatrix} \frac{3}{2}k & -k \\ -k & \frac{2}{3}k \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \;=\; 0\ ,$$

hence $\mathbf{e}_-$ is proportional to $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

3. A particle of mass m moves along a line subject to the potential energy

$$U(x) = \frac{1}{2}kx^2 + \frac{\lambda}{6}x^6$$

where $\lambda > 0$. We are interested in the oscillatory motion of amplitude A.

(a) Find an expression for the period of oscillation T as a definite integral. (You need not attempt to evaluate this integral!)

(b) Use perturbation theory to find the motion $x(t)$ with initial conditions $x(0) = A$, $\dot{x}(0) = 0$ through first order in λ, where λ is considered "small". [*Hint:* $\cos^5\theta = \frac{1}{16}\cos 5\theta + \frac{5}{16}\cos 3\theta + \frac{5}{8}\cos\theta$.] Explain what a "secular term" is, and say which term in your answer is the secular term.

(c) Use the Lindstedt renormalization procedure to compute the frequency of oscillation ω through first order in λ.

SIMILAR SEEN: (a) 5 points, (b) 12 points, (c) 8 points

SOLUTION:

(a) Energy conservation gives

$$\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + U(x) = E \,,$$

so that

$$\frac{dx}{dt} = \pm\sqrt{\frac{2}{m}[E - U(x)]}$$

and hence

$$\frac{dx}{\sqrt{\frac{2}{m}[E - U(x)]}} = dt \,.$$

So one half-period of oscillation corresponds to the motion from $x = -A$ to $x = +A$, when the total energy equals $U(A) = \frac{1}{2}kA^2 + \frac{\lambda}{6}A^6$, hence

$$\boxed{T = 2\int_{-A}^{A} \frac{dx}{\sqrt{\frac{2}{m}\left[\frac{1}{2}kA^2 + \frac{\lambda}{6}A^6 - \frac{1}{2}kx^2 - \frac{\lambda}{6}x^6\right]}}}$$

(b) We write

$$x(t) \; = \; x_0(t) \, + \, \lambda x_1(t) \, + \, \lambda^2 x_2(t) \, + \, \ldots$$

and insert this into the differential equation

$$m\ddot{x} \; = \; -kx \, - \, \lambda x^5$$

with initial conditions

$$\begin{aligned} x(0) &= A + 0\lambda + 0\lambda^2 + \ldots \\ \dot{x}(0) &= 0 + 0\lambda + 0\lambda^2 + \ldots \end{aligned}$$

Collecting terms of order λ^0, we find the differential equation

$$m\ddot{x}_0 \; = \; -kx_0$$

with initial condition $x_0(0) = A$, $\dot{x}_0(0) = 0$, so the solution is

$$\boxed{x_0(t) \; = \; A\cos\omega_0 t}$$

where $\omega_0 = \sqrt{k/m}$.

Collecting terms of order λ^1, we find the differential equation

$$m\ddot{x}_1 \; = \; -kx_1 \, - \, x_0^5$$

or in other words

$$\begin{aligned} m\ddot{x}_1 + kx_1 &= -A^5 \cos^5 \omega_0 t \\ &= -\frac{1}{16} A^5 \cos 5\omega_0 t \, - \, \frac{5}{16} A^5 \cos 3\omega_0 t \, - \, \frac{5}{8} A^5 \cos \omega_0 t \end{aligned}$$

where we used the trig identity $\cos^5\theta = \frac{1}{16}\cos 5\theta + \frac{5}{16}\cos 3\theta + \frac{5}{8}\cos\theta$. The initial condition is $x_1(0) = 0$, $\dot{x}_1(0) = 0$. We find particular solutions corresponding to the two forcing terms on the right-hand side (forcing at the resonant frequency ω_0 leads to a response $t\sin\omega_0 t$) and add in a solution to the homogeneous equation so as to satisfy the two initial conditions; the result is

$$\boxed{x_1(t) \; = \; \frac{A^5}{m\omega_0^2}\left(\frac{1}{384}\cos 5\omega_0 t \, + \, \frac{5}{128}\cos 3\omega_0 t \, - \, \frac{1}{24}\cos\omega_0 t \, - \, \frac{5}{16}\omega_0 t \sin\omega_0 t\right)}$$

A "secular term" is a term that is not periodic, but rather *grows* in amplitude as $t \to +\infty$ and/or $t \to -\infty$. The term $t\sin\omega_0 t$ in $x_1(t)$ is a secular term.

(c) We assume that actual frequency of oscillation ω, which of course depends on λ, can be expanded as a power series in λ:

$$\omega \;=\; \omega_0 + \lambda\omega_1 + \lambda^2\omega_2 + \ldots$$

with some unknown coefficients $\omega_1, \omega_2, \ldots$. We then turn the above equation around,

$$\omega_0 \;=\; \omega - \lambda\omega_1 - \lambda^2\omega_2 - \ldots\,,$$

and insert it into the force law; since $k = m\omega_0^2$, the equation of motion becomes

$$m\ddot{x} + m(\omega - \lambda\omega_1 - \lambda^2\omega_2 - \ldots)^2 x \;=\; -\lambda x^3\,.$$

We then carry out perturbation theory as before, comparing coefficients of each power of λ and working our way upwards:

Collecting terms of order λ^0, we find the differential equation

$$m\ddot{x}_0 + m\omega^2 x_0 \;=\; 0$$

with initial condition $x_0(0) = A$, $\dot{x}_0(0) = 0$, so the solution is

$$\boxed{x_0(t) \;=\; A\cos\omega t}$$

Note that the frequency here is the (as-yet-unknown) *true* frequency ω.

Collecting terms of order λ^1, we find the differential equation

$$m\ddot{x}_1 + m\omega^2 x_1 \;=\; -x_0^3 + 2m\omega\omega_1 x_0$$

or in other words

$$m\ddot{x}_1 + m\omega^2 x_1 \;=\; -\frac{1}{16}A^5\cos 5\omega t - \frac{5}{16}A^5\cos 3\omega t - \left(\frac{5}{8}A^5 - 2m\omega\omega_1\right)\cos\omega t\,.$$

The initial condition is $x_1(0) = 0$, $\dot{x}_1(0) = 0$. We can eliminate the secular term by choosing

$$\boxed{\omega_1 \;=\; \frac{5A^4}{16m\omega}}$$

Therefore

$$\begin{aligned}\omega \;&=\; \omega_0 + \lambda\omega_1 + O(\lambda^2)\\ &=\; \omega_0 + \lambda\frac{5A^4}{16m\omega} + O(\lambda^2)\\ &=\; \omega_0 + \lambda\frac{5A^4}{16m[\omega_0 + O(\lambda)]} + O(\lambda^2)\\ &=\; \omega_0\left[1 + \frac{5}{16}\frac{\lambda A^4}{m\omega_0^2} + O(\lambda^2)\right].\end{aligned}$$

4. A particle of mass m slides under the influence of gravity on the frictionless inner surface of a hemispherical bowl of radius R. Use cylindrical coordinates (r, φ, z) with z in the vertical direction.

   (a) Write the r, φ and z components of Newton's equations of motion for the bead. Your equations will contain an unknown constraint force.

   (b) Find the equations of motion for the bead coordinates (r, φ), by eliminating z and the constraint force.

   (c) Find the Lagrangian and obtain the equations of motion for the bead coordinates (r, φ). Should your answer agree with the one from part (b)? Explain why or why not.

   (d) Find two conserved quantities.

   (e) Find a closed equation of motion for r alone. One of the conserved quantities will appear as a parameter in your equation.

   (f) Find the speed v_0 at which the bead will move in a horizontal circle of radius r_0.

   (g) Find the frequency of small radial oscillations around the circular motion found in part (f).

SIMILAR SEEN: (a) 4 points, (b) 3 points, (c) 6 points, (d) 3 points, (e) 3 points, (f) 2 points, (g) 4 points

SOLUTION:

(a) There are two forces acting on the bead: the gravitational force mg downwards, and the normal force exerted by the surface on the bead. If we put $z = 0$ at the top of the bowl, the equation of the hemisphere is $z^2 + r^2 = R^2$, and the normal vector points inwards towards the origin, i.e. the unit normal vector is $(-z/R)\widehat{\mathbf{e}}_z + (-r/R)\widehat{\mathbf{e}}_r$. Therefore the normal force is

$$\mathbf{N} \;=\; (-z/R)\, N\, \widehat{\mathbf{e}}_z \;-\; (-r/R)\, N\, \widehat{\mathbf{e}}_r$$

where N is the unknown (and in general time-dependent) magnitude of the normal force. Using the known forms for acceleration in plane polar (and hence cylindrical) coordinates, we have

$$\begin{aligned} m(\ddot{r} - r\dot{\varphi}^2) &= -\frac{r}{R} N \\ m(r\ddot{\varphi} + 2\dot{r}\dot{\varphi}) &= 0 \\ m\ddot{z} &= -mg - \frac{z}{R} N \end{aligned}$$

for the r, φ and z components of $\mathbf{F} = m\mathbf{a}$, respectively.

(b) Eliminate N by adding $-r/z = r/\sqrt{R^2 - r^2}$ times the z equation from part (a) to the r equation:

$$m\Big(\ddot{r} - r\dot{\varphi}^2 - \frac{r}{z}\ddot{z}\Big) \;=\; -mg\frac{r}{z}\;.$$

Now use $z = -\sqrt{R^2 - r^2}$ to eliminate z: we find

$$\dot{z} \;=\; \frac{r}{\sqrt{R^2 - r^2}}\,\dot{r}$$

and then

$$\ddot{z} \;=\; \frac{r}{\sqrt{R^2 - r^2}}\,\ddot{r} \;+\; \frac{R^2}{(R^2 - r^2)^{3/2}}\,\dot{r}^2\;.$$

Inserting this into the equation of motion, we find (after division by m)

$$\boxed{\frac{R^2}{R^2 - r^2}\ddot{r} \;+\; \frac{R^2}{(R^2 - r^2)^2}r\dot{r}^2 \;-\; r\dot{\varphi}^2 \;+\; \frac{gr}{\sqrt{R^2 - r^2}} \;=\; 0}$$

(c) If we put $z = 0$ at the top of the bowl, the equation of the hemisphere is $z = -\sqrt{R^2 - r^2}$. The kinetic energy is therefore

$$\begin{aligned}
T &= \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2 + \dot{z}^2) \\
&= \tfrac{1}{2}m\Big(\dot{r}^2 + r^2\dot{\varphi}^2 + \frac{r^2}{R^2 - r^2}\dot{r}^2\Big) \\
&= \tfrac{1}{2}m\Big(\frac{R^2}{R^2 - r^2}\dot{r}^2 + r^2\dot{\varphi}^2\Big)
\end{aligned}$$

and the potential energy is

$$U \;=\; mgz \;=\; -mg\sqrt{R^2 - r^2}\;,$$

so the Lagrangian $L(r, \varphi, \dot{r}, \dot{\varphi})$ is

$$L \;=\; T - U \;=\; \tfrac{1}{2}m\Big[\frac{R^2}{R^2 - r^2}\dot{r}^2 \;+\; r^2\dot{\varphi}^2\Big] \;+\; mg\sqrt{R^2 - r^2}\;.$$

Now we start differentiating. First let's do the r equation:

$$\frac{\partial L}{\partial \dot{r}} \;=\; \frac{mR^2}{R^2 - r^2}\dot{r}$$

so that

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{r}}\Big) \;=\; \frac{mR^2}{R^2 - r^2}\ddot{r} \;+\; \frac{2mR^2}{(R^2 - r^2)^2}r\dot{r}^2\;.$$

On the other hand,

$$\frac{\partial L}{\partial r} \;=\; \frac{mR^2}{(R^2 - r^2)^2}r\dot{r}^2 \;+\; mr\dot{\varphi}^2 \;-\; \frac{mgr}{\sqrt{R^2 - r^2}}\;.$$

Therefore, Lagrange's equation of motion $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$ is (after division by m)

$$\boxed{\frac{R^2}{R^2 - r^2}\ddot{r} \;+\; \frac{R^2}{(R^2 - r^2)^2} r\dot{r}^2 \;-\; r\dot{\varphi}^2 \;+\; \frac{gr}{\sqrt{R^2 - r^2}} \;=\; 0}$$

which agrees with part (b) — as it should, because the Newtonian and Lagrangian methods are equivalent.

Now let's do the φ equation:

$$\frac{\partial L}{\partial \dot{\varphi}} \;=\; mr^2 \dot{\varphi} \;,$$

so that

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) \;=\; mr^2 \ddot{\varphi} \;+\; 2mr\dot{r}\dot{\varphi} \;.$$

On the other hand,

$$\frac{\partial L}{\partial \varphi} \;=\; 0 \;.$$

So Lagrange's equation of motion $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) - \frac{\partial L}{\partial \varphi} = 0$ is (after division by mr)

$$\boxed{r\ddot{\varphi} + 2\dot{r}\dot{\varphi} \;=\; 0}$$

which again agrees (as it should) with part (b).

(d) The coordinate φ is a cyclic coordinate, i.e. $\frac{\partial L}{\partial \varphi} = 0$. So the corresponding conjugate momentum, $p_\varphi = \partial L / \partial \dot{\varphi} = mr^2 \dot{\varphi}$, is conserved. This is the z component of the angular momentum.

Since the Lagrangian has no explicit time dependence, the "Hamiltonian" $h = p_r \dot{r} + p_\varphi \dot{\varphi} - L$ is conserved. And since this is a conservative standard system — that is, the kinetic energy T is quadratic in the velocities, and the potential energy U is independent of the velocities — the "Hamiltonian" h simply equals the total energy $E = T + U$:

$$E \;=\; \tfrac{1}{2} m \left[\frac{R^2}{R^2 - r^2}\dot{r}^2 \;+\; r^2 \dot{\varphi}^2\right] \;-\; mg\sqrt{R^2 - r^2} \;.$$

(e) As in the central-force problem, we exploit conservation of angular momentum by writing $\dot{\varphi} = \ell / mr^2$ where ℓ is the value of the angular momentum, then inserting this into the r equation of motion to get an equation for r alone:

$$\boxed{\frac{R^2}{R^2 - r^2}\ddot{r} \;+\; \frac{R^2}{(R^2 - r^2)^2} r\dot{r}^2 \;-\; \frac{\ell^2}{m^2 r^3} \;+\; \frac{gr}{\sqrt{R^2 - r^2}} \;=\; 0}$$

(f) If the bead moves in a horizontal circle of radius r_0, we have simply $r = r_0 =$ constant, independent of time. In this case $\dot{r} = 0$ and $\ddot{r} = 0$, so the preceding equation of motion gives

$$\frac{\ell^2}{m^2 r_0^3} \;=\; \frac{g r_0}{\sqrt{R^2 - r_0^2}} \; .$$

Since $\ell = m r_0 v_0$, we have

$$\boxed{v_0 \;=\; \sqrt{g}\, r_0 \,(R^2 - r_0^2)^{-1/4}}$$

(This result is also easy to obtain by elementary Newtonian considerations.)

(g) To study small oscillations around the solution $r = r_0 =$ constant, write $r = r_0 + \epsilon(t)$, insert into the equation of motion found in part (e), and expand in Taylor series in ϵ, *retaining only the linear terms in* ϵ. We obtain

$$\frac{R^2}{R^2 - r_0^2}\ddot{\epsilon} \;+\; \left(\frac{3\ell^2}{m^2 r_0^4} \;+\; \frac{g R^2}{(R^2 - r_0^2)^{3/2}}\right)\epsilon \;=\; 0 \; .$$

But from part (f) we have

$$\frac{\ell^2}{m^2} \;=\; \frac{g r_0^4}{(R^2 - r_0^2)^{1/2}} \; ,$$

hence

$$\frac{R^2}{R^2 - r_0^2}\ddot{\epsilon} \;+\; \frac{4R^2 - 3r_0^2}{(R^2 - r_0^2)^{3/2}} g\epsilon \;=\; 0 \; .$$

The frequency of small oscillations is therefore

$$\boxed{\omega_0 \;=\; \frac{[(4R^2 - 3r_0^2)^{1/2} g^{1/2}}{R\,(R^2 - r_0^2)^{1/4}}}$$

5. A particle of mass m moves on a smooth horizontal table. It is connected to a massless inextensible string that passes through a small hole in the table, and the string is pulled from below in such a way that the particle's distance from the hole is a specified function $R(t)$. Use polar coordinates (r, θ) with the origin located at the hole.

   (a) Using θ as the generalised coordinate, find the kinetic energy, the potential energy, and the Lagrangian.

   (b) Show that θ is a cyclic coordinate, and find the corresponding conserved momentum p_θ. What is the physical meaning of p_θ?

   (c) Find the Hamiltonian and the Hamilton equations of motion.

   (d) Compare the Hamiltonian and the total energy. Is the Hamiltonian conserved? Is the total energy conserved? Justify your answers, and explain physically.

UNSEEN: (a) 8 points, (b) 4 points, (c) 7 points, (d) 6 points

SOLUTION:

(a) Use cylindrical coordinates (r, θ). The constraint tells us that $r = R(t)$. So the kinetic energy is

$$T \;=\; \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) \;=\; \frac{1}{2}m[\dot{R}(t)^2 + R(t)^2\dot{\theta}^2] \;.$$

The potential energy is $U = 0$. Therefore the Lagrangian $L(\theta, \dot{\theta})$ is

$$L \;=\; T - U \;=\; \frac{1}{2}m[\dot{R}(t)^2 + R(t)^2\dot{\theta}^2] \;.$$

(b) The conjugate momentum p_θ is

$$p_\theta \;=\; \frac{\partial L}{\partial \dot{\theta}} \;=\; mR(t)^2\dot{\theta} \;,$$

which is the angular momentum of the particle. The coordinate θ is cyclic because $\partial L/\partial\theta = 0$, so p_θ is conserved.

(c) We have $\dot{\theta} = p_\theta / mR(t)^2$. Then the Hamiltonian $H(\theta, p_\theta)$ is

$$\begin{aligned} H &= p_\theta \dot{\theta} - L \\ &= mR(t)^2\dot{\theta}^2 - \frac{1}{2}m\dot{R}(t)^2 - \frac{1}{2}mR(t)^2\dot{\theta}^2 \\ &= \frac{1}{2}mR(t)^2\dot{\theta}^2 - \frac{1}{2}m\dot{R}(t)^2 \\ &= \frac{p_\theta^2}{2mR(t)^2} - \frac{1}{2}m\dot{R}(t)^2 \, . \end{aligned}$$

The Hamilton equations of motion are then

$$\begin{aligned} \dot{p}_\theta &= -\frac{\partial H}{\partial \theta} = 0 \\ \dot{\theta} &= \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mR(t)^2} \end{aligned}$$

which agree with what we know from the Lagrangian formulation.

(d) The total energy is

$$\begin{aligned} E = T &= \frac{1}{2}mR(t)^2\dot{\theta}^2 + \frac{1}{2}m\dot{R}(t)^2 \\ &= \frac{p_\theta^2}{2mR(t)^2} + \frac{1}{2}m\dot{R}(t)^2 \, , \end{aligned}$$

while the Hamiltonian is

$$\begin{aligned} H &= \frac{1}{2}mR(t)^2\dot{\theta}^2 - \frac{1}{2}m\dot{R}(t)^2 \\ &= \frac{p_\theta^2}{2mR(t)^2} - \frac{1}{2}m\dot{R}(t)^2 \\ &= E - m\dot{R}(t)^2 \, . \end{aligned}$$

In general neither is conserved, unless $R(t) =$ constant. Physically this is because the tension in the string does work on the particle.

6. Consider the Lagrangian

$$L \;=\; \tfrac{1}{2}A(\dot\theta^2 + \dot\varphi^2\sin^2\theta) \;+\; \tfrac{1}{2}B(\dot\psi + \dot\varphi\cos\theta)^2 \;-\; C\cos\theta$$

where θ, φ, ψ are the generalised coordinates, and A, B, C are positive constants.

(a) Find the three generalised momenta p_θ, p_φ and p_ψ.

(b) Use Lagrange's equations to show that p_φ and p_ψ are constants of motion.

(c) Find the Hamiltonian $H(\theta, \varphi, \psi, p_\theta, p_\varphi, p_\psi)$, and use Hamilton's equations to show that p_φ and p_ψ are constants of motion.

(d) Deduce that the Hamiltonian can be written in the form

$$H \;=\; \frac{p_\theta^2}{2A} \;+\; U(\theta)\;,$$

and find the "effective potential" $U(\theta)$ [in which p_φ and p_ψ will appear as parameters].

(e) Use Hamilton's equations to find the equation of motion expressing $\ddot\theta$ in terms of $U(\theta)$.

(f) The system is started with initial conditions $\theta(0) = \theta_0$ and $\dot\theta(0) = 0$, where θ_0 is a local minimum of $U(\theta)$. Find the subsequent motion of θ, φ and ψ.

UNSEEN: (a) 4 points, (b) 4 points, (c) 7 points, (d) 3 points, (e) 3 points, (f) 4 points

SOLUTION:

(a) By definition we have

$$p_\theta \;=\; \frac{\partial L}{\partial\dot\theta} \;=\; A\dot\theta$$

$$\begin{aligned} p_\varphi \;&=\; \frac{\partial L}{\partial\dot\varphi} \;=\; A\sin^2\theta\,\dot\varphi \;+\; B\cos\theta(\dot\psi + \dot\varphi\cos\theta) \\ &=\; (A\sin^2\theta + B\cos^2\theta)\dot\varphi \;+\; B\cos\theta\,\dot\psi \end{aligned}$$

$$p_\psi \;=\; \frac{\partial L}{\partial\dot\psi} \;=\; B(\dot\psi + \dot\varphi\cos\theta)$$

(b) Since $\partial L/\partial\varphi = 0$, Lagrange's equations tell us that p_φ is a constant of motion. Likewise, since $\partial L/\partial\psi = 0$, Lagrange's equations tell us that p_ψ is a constant of motion.

(c) We first compute the generalised velocities in terms of the conjugate momenta:

$$
\begin{aligned}
\dot\theta &= \frac{p_\theta}{A} \\
\dot\psi + \dot\varphi\cos\theta &= \frac{p_\psi}{B} \\
\dot\varphi &= \frac{p_\varphi \,-\, B\cos\theta\dfrac{p_\psi}{B}}{A\sin^2\theta} \;=\; \frac{p_\varphi - p_\psi\cos\theta}{A\sin^2\theta} \\
\dot\psi &= \frac{p_\psi}{B} \,-\, \dot\varphi\cos\theta \\
&= \frac{p_\psi}{B} \,-\, \frac{(p_\varphi - p_\psi\cos\theta)\cos\theta}{A\sin^2\theta}
\end{aligned}
$$

Then the Hamiltonian $H(\theta,\varphi,\psi,p_\theta,p_\varphi,p_\psi)$ is

$$
\begin{aligned}
H &= \dot\theta p_\theta \,+\, \dot\varphi p_\varphi \,+\, \dot\psi p_\psi \,-\, L \\
&= \frac{p_\theta^2}{A} \,+\, \frac{p_\varphi \,-\, p_\psi\cos\theta}{A\sin^2\theta}p_\varphi \,+\, \left(\frac{p_\psi}{B} \,-\, \frac{(p_\varphi - p_\psi\cos\theta)\cos\theta}{A\sin^2\theta}\right)p_\psi \\
&\qquad -\frac{A}{2}\left(\frac{p_\theta^2}{A^2} \,+\, \frac{(p_\varphi - p_\psi\cos\theta)^2}{A^2\sin^2\theta}\right) \,-\, \frac{B}{2}\left(\frac{p_\psi}{B}\right)^2 \,+\, C\cos\theta \\
&= \frac{p_\theta^2}{2A} \,+\, \frac{p_\psi^2}{2B} \,+\, \frac{(p_\varphi - p_\psi\cos\theta)^2}{2A\sin^2\theta} \,+\, C\cos\theta\;.
\end{aligned}
$$

Since $\partial H/\partial\varphi = 0$, Hamilton's equations tell us that p_φ is a constant of motion. Likewise, since $\partial H/\partial\psi = 0$, Hamilton's equations tell us that p_ψ is a constant of motion.

(d) It follows immediately that

$$H \;=\; \frac{p_\theta^2}{2A} \,+\, U(\theta)$$

where

$$U(\theta) \;=\; \frac{p_\psi^2}{2B} \,+\, \frac{(p_\varphi - p_\psi\cos\theta)^2}{2A\sin^2\theta} \,+\, C\cos\theta\;.$$

[The first term in $U(\theta)$ is an unimportant constant; the second and third ones are the crucial ones because they depend on θ.]

(e) Using the result of part (d), Hamilton's equations $\dot{\theta} = \partial H/\partial p_\theta$ and $\dot{p}_\theta = -\partial H/\partial \theta$ imply that

$$A\ddot{\theta} = -U'(\theta) \; .$$

(f) Since $U'(\theta_0) = 0$, the equation from part (e) together with the initial conditions $\theta(0) = \theta_0$ and $\dot{\theta}(0) = 0$ imply that $\theta(t) = \theta_0$ for all t. Then $\dot{\varphi}$ and $\dot{\psi}$ are constant in time as well (at values given by the formulae in the answer to (c) together with the given values of p_φ, p_ψ and θ_0).