Do **all** *problems.* **All problems will count.**

Show all your work. Make sure that you have adequately explained the method you have used to solve the problem.

1. A chain of linear density (mass per unit length) σ lies in a heap on a frictionless horizontal surface. You grab an end and pull horizontally with a constant force F, starting at time 0. Assume that the chain has no friction with itself.

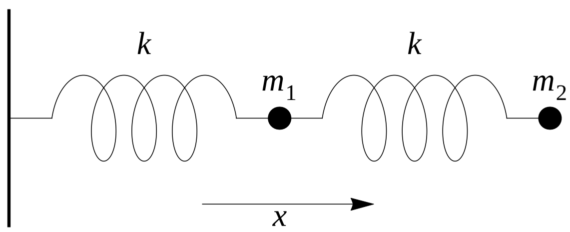
   (a) Find the position x of the end of the chain, as a function of time t, during the time that the chain is unraveling. Explain carefully what principle you used to obtain this answer.

   (b) What is the kinetic energy of the chain at time t?

   (c) Between time 0 and time t, how much work have you done?

   (d) Compare your answers to (b) and (c). If they are the same, explain why; and if one of them is larger than the other, also explain why.

2. Two particles, of masses m_1 and m_2, respectively, are connected to a fixed wall by springs of spring constant k, as shown in the diagram:



   The particles move horizontally. Let x_1 and x_2 be the displacements of the two particles from their equilibrium positions.

   (a) Derive the equations of motion. (You may use either Newtonian or Lagrangian methods.)

   (b) Find the frequencies of the normal modes.

   (c) In the special case $m_1/m_2 = 3/2$, find the eigenvectors corresponding to the normal modes.

3. A particle of mass m moves along a line subject to the potential energy

$$U(x) = \frac{1}{2}kx^2 + \frac{\lambda}{6}x^6$$

where $\lambda > 0$. We are interested in the oscillatory motion of amplitude A.

(a) Find an expression for the period of oscillation T as a definite integral. (You need not attempt to evaluate this integral!)

(b) Use perturbation theory to find the motion $x(t)$ with initial conditions $x(0) = A$, $\dot{x}(0) = 0$ through first order in λ, where λ is considered "small". [*Hint:* $\cos^5\theta = \frac{1}{16}\cos 5\theta + \frac{5}{16}\cos 3\theta + \frac{5}{8}\cos\theta$.] Explain what a "secular term" is, and say which term in your answer is the secular term.

(c) Use the Lindstedt renormalization procedure to compute the frequency of oscillation ω through first order in λ.

4. A particle of mass m slides under the influence of gravity on the frictionless inner surface of a hemispherical bowl of radius R. Use cylindrical coordinates (r, φ, z) with z in the vertical direction.

(a) Write the r, φ and z components of Newton's equations of motion for the bead. Your equations will contain an unknown constraint force.

(b) Find the equations of motion for the bead coordinates (r, φ), by eliminating z and the constraint force.

(c) Find the Lagrangian and obtain the equations of motion for the bead coordinates (r, φ). Should your answer agree with the one from part (b)? Explain why or why not.

(d) Find two conserved quantities.

(e) Find a closed equation of motion for r alone. One of the conserved quantities will appear as a parameter in your equation.

(f) Find the speed v_0 at which the bead will move in a horizontal circle of radius r_0.

(g) Find the frequency of small radial oscillations around the circular motion found in part (f).

5. A particle of mass m moves on a smooth horizontal table. It is connected to a massless inextensible string that passes through a small hole in the table, and the string is pulled from below in such a way that the particle's distance from the hole is a specified function $R(t)$. Use polar coordinates (r, θ) with the origin located at the hole.

   (a) Using θ as the generalised coordinate, find the kinetic energy, the potential energy, and the Lagrangian.

   (b) Show that θ is a cyclic coordinate, and find the corresponding conserved momentum p_θ. What is the physical meaning of p_θ?

   (c) Find the Hamiltonian and the Hamilton equations of motion.

   (d) Compare the Hamiltonian and the total energy. Is the Hamiltonian conserved? Is the total energy conserved? Justify your answers, and explain physically.

6. Consider the Lagrangian

$$L = \tfrac{1}{2}A(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta) + \tfrac{1}{2}B(\dot{\psi} + \dot{\varphi}\cos\theta)^2 - C\cos\theta$$

   where θ, φ, ψ are the generalised coordinates, and A, B, C are positive constants.

   (a) Find the three generalised momenta p_θ, p_φ and p_ψ.

   (b) Use Lagrange's equations to show that p_φ and p_ψ are constants of motion.

   (c) Find the Hamiltonian $H(\theta, \varphi, \psi, p_\theta, p_\varphi, p_\psi)$, and use Hamilton's equations to show that p_φ and p_ψ are constants of motion.

   (d) Deduce that the Hamiltonian can be written in the form

$$H = \frac{p_\theta^2}{2A} + U(\theta) ,$$

   and find the "effective potential" $U(\theta)$ [in which p_φ and p_ψ will appear as parameters].

   (e) Use Hamilton's equations to find the equation of motion expressing $\ddot{\theta}$ in terms of $U(\theta)$.

   (f) The system is started with initial conditions $\theta(0) = \theta_0$ and $\dot{\theta}(0) = 0$, where θ_0 is a local minimum of $U(\theta)$. Find the subsequent motion of θ, φ and ψ.