

MATHEMATICS 0054 (Analytical Dynamics)
YEAR 2021–2022, TERM 2

PROBLEM SET #7 (last one!!!)

This problem set is due at the *beginning* of lecture on Wednesday 23 March.

Topics: Hamiltonian approach to mechanics: phase space, Hamilton's equations, Liouville's theorem, Poisson brackets.

Reading:

- Gregory, *Classical Mechanics*, Chapter 14 (handout).
- Handout #12: The Hamiltonian approach to mechanics.

1. *Spherical pendulum.* A particle of mass m is attached to a massless inextensible string of length ℓ and hung from the ceiling in a uniform gravitational field g . The pendulum is free to move in three dimensions, i.e. not necessarily in a fixed plane. Use spherical coordinates with the north pole pointing downwards, i.e. θ is the angle that the string makes with the vertical, and φ is the azimuthal angle.
 - (a) Using the generalized coordinates (θ, φ) , find the Lagrangian and Lagrange's equations of motion. Identify any cyclic coordinates and interpret the conserved conjugate momenta.
 - (b) Find the Hamiltonian and Hamilton's equations of motion. Once again identify any cyclic coordinates and interpret the conserved conjugate momenta.
2. [An old friend: See Problem 3 of Problem Set #5]

A smooth thin wire is bent into the shape of a parabola, $z = x^2/2a$, and is made to rotate with angular velocity ω about the z axis [i.e. about the point $x = 0$ on the wire]; here the $+z$ direction is of course oriented upwards. A bead of mass m then slides frictionlessly on the wire under the influence of gravity. Use cylindrical coordinates (r, φ, z) .

Find the Lagrangian in terms of the generalized coordinate r , and then find the Hamiltonian. Is H equal to the total energy? Is H conserved?

3. Consider a free particle in a curvilinear coordinate system $\{q_\alpha\}$. The Lagrangian is $L = T$, and the Lagrange equations of motion are

$$\dot{p}_\alpha = \frac{\partial T}{\partial q_\alpha}.$$

The Hamiltonian is $H = T$, and the Hamilton equations of motion are

$$\dot{p}_\alpha = -\frac{\partial T}{\partial q_\alpha}.$$

How are these two formulae for \dot{p}_α to be reconciled? Illustrate your answer by considering the case of plane polar coordinates.

4. [Another old friend: See Problem 3 of Problem Set #6]

Recall that the Lagrangian for a particle with electric charge e moving in an electromagnetic field is

$$L(\mathbf{r}, \dot{\mathbf{r}}, t) = \frac{1}{2}m\dot{\mathbf{r}}^2 - e\varphi(\mathbf{r}, t) + e\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t)$$

where $\mathbf{A}(\mathbf{r}, t)$ is the vector potential and $\varphi(\mathbf{r}, t)$ is the scalar potential.

- Find the conjugate momentum \mathbf{p} in terms of the positions and velocities. Is \mathbf{p} the ordinary linear momentum?
 - Find the Hamiltonian $H(\mathbf{r}, \mathbf{p}, t)$.
 - Find Hamilton's equations of motion, and show that they are equivalent to Lagrange's equations of motion.
 - Under what circumstances is H conserved?
5. Let $\mathbf{q} = (q_1, \dots, q_n)$ and $\mathbf{p} = (p_1, \dots, p_n)$ be canonical coordinates. Recall that the Poisson bracket of two functions $f(\mathbf{q}, \mathbf{p})$ and $g(\mathbf{q}, \mathbf{p})$ is defined as

$$\{f, g\} = \sum_{j=1}^n \left(\frac{\partial f}{\partial q_j} \frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q_j} \right).$$

- Show that the Poisson bracket satisfies $\{fg, h\} = f\{g, h\} + \{f, h\}g$ for any three functions $f(\mathbf{q}, \mathbf{p})$, $g(\mathbf{q}, \mathbf{p})$, $h(\mathbf{q}, \mathbf{p})$.

For the remainder of this problem, suppose that $\mathbf{q} = (q_1, q_2, q_3)$ are Cartesian coordinates for a single particle, and that $\mathbf{p} = (p_1, p_2, p_3)$ is the particle's momentum.

- Express the angular momentum \mathbf{L} in terms of \mathbf{q} and \mathbf{p} , and compute the Poisson brackets $\{q_i, L_j\}$ and $\{p_i, L_j\}$. You may wish to express your answers in terms of the antisymmetric symbol ϵ_{ijk} , defined as

$$\begin{aligned} \epsilon_{123} &= \epsilon_{231} = \epsilon_{312} = +1 \\ \epsilon_{132} &= \epsilon_{321} = \epsilon_{213} = -1 \\ \epsilon_{ijk} &= 0 \quad \text{if } i, j, k \text{ are not all distinct} \end{aligned}$$

- (c) Show that $\{L_i, L_j\} = \sum_{k=1}^3 \epsilon_{ijk} L_k$, and write out explicitly what this means in terms of L_1, L_2, L_3 . [*Hint*: The identity $\sum_{i=1}^3 \epsilon_{ijk} \epsilon_{ilm} = \delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}$ may be useful. Can you prove it?] [*Remark*: In formulae like these, it is often convenient to use the **Einstein summation convention**, which says that repeated indices are automatically summed (in this case from 1 to 3). So this identity would be written simply as $\epsilon_{ijk} \epsilon_{ilm} = \delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}$, and the equation we want to prove would be written as $\{L_i, L_j\} = \epsilon_{ijk} L_k$.]
- (d) Show that $\{L_i, |\mathbf{L}|^2\} = 0$.