MATHEMATICS 0054 (Analytical Dynamics)
YEAR 2019–2020, TERM 2

PROBLEM SETS #6+7 (CONDENSED)

Because of the coronavirus situation, this problem set will *not* be assessed: I am giving it to you to help you practice for the final exam. A week or two from now I will post the solutions on the course website, but I strongly urge you to struggle with the problems *before* consulting my solutions.

Note also:

1) The topic of "symmetries and conservation laws" (Handout #11, section 4) is very beautiful and I very much hope you will study it, but it will *not* be included on the final exam, and I have *not* included any problems on it in this problem set. Likewise for the topic of "variational principles" (Gregory, Chapter 13), which I had hoped to teach you but have now chosen to omit, for lack of time.

2) For the Hamiltonian approach to mechanics, you should study Gregory, Chapter 14 (posted on the course website) and then attempt the problems in this problem set. I am also posting Handout #12, which covers the beautiful topics of Poisson brackets and canonical transformations, but this material will *not* be included on the final exam, and I have *not* included any problems on it in this problem set.

Please do e-mail me with your questions concerning the handouts and the problems. We will probably not be having face-to-face classes, but I remain available to assist you in any way I can. By e-mail we can also make an appointment for a telephone consultation.

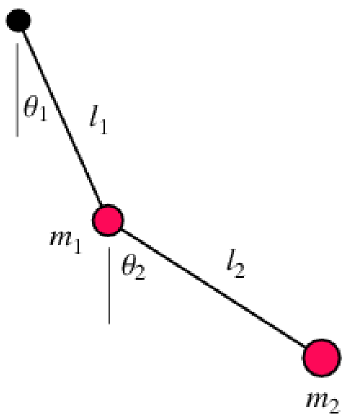
Topics: Lagrangian approach to mechanics, continued. Hamiltonian approach to mechanics.

Reading:

- Gregory, *Classical Mechanics*, Chapter 12 (handout).
- Handout #11: The Lagrangian approach to mechanics.
- Gregory, *Classical Mechanics*, Chapter 14 (handout).
- Handout #12: The Hamiltonian approach to mechanics.

1. [An old friend: See Problem 3 of Problem Set #3]

   A double pendulum consists of rigid massless rods of lengths ℓ_1 and ℓ_2 and particles of mass m_1 and m_2, respectively, attached as in the diagram. (All pivots are frictionless, of course.)

   (a) Find the Lagrangian for the system, using as generalized coordinates the angles θ_1 and θ_2.

   (b) Find the exact equations of motion for the system, using the Lagrangian. Do the equations of motion agree with those found by Newtonian methods in Problem Set #3?

2. *Spherical pendulum.* A particle of mass m is attached to a massless inextensible string of length ℓ and hung from the ceiling in a uniform gravitational field g. The pendulum is free to move in three dimensions, i.e. not necessarily in a fixed plane. Use spherical coordinates with the north pole pointing downwards, i.e. θ is the angle that the string makes with the vertical, and φ is the azimuthal angle.

   (a) Using the generalized coordinates (θ, φ), find the Lagrangian and Lagrange's equations of motion. Identify any cyclic coordinates and interpret the conserved conjugate momenta.

   (b) Find the Hamiltonian and Hamilton's equations of motion. Once again identify any cyclic coordinates and interpret the conserved conjugate momenta.

3. [An old friend: See Problem 3 of Problem Set #5]

   A smooth thin wire is bent into the shape of a parabola, $z = x^2/2a$, and is made to rotate with angular velocity ω about the z axis [i.e. about the point $x = 0$ on the wire]; here the $+z$ direction is of course oriented upwards. A bead of mass m then slides frictionlessly on the wire under the influence of gravity. Use cylindrical coordinates (r, φ, z).

   Find the Lagrangian in terms of the generalized coordinate r, and then find the Hamiltonian. Is H equal to the total energy? Is H conserved?

4. Consider a free particle in a curvilinear coordinate system $\{q_\alpha\}$. The Lagrangian is $L = T$, and the Lagrange equations of motion are

$$\dot{p}_\alpha = \frac{\partial T}{\partial q_\alpha} \, .$$

The Hamiltonian is $H = T$, and the Hamilton equations of motion are

$$\dot{p}_\alpha = -\frac{\partial T}{\partial q_\alpha} \, .$$

How are these two formulae for $\dot{p}_\alpha$ to be reconciled? Illustrate your answer by considering the case of plane polar coordinates.

5. (a) Show that the Lagrangian function

$$L(\mathbf{r}, \dot{\mathbf{r}}, t) = \frac{1}{2} m \dot{\mathbf{r}}^2 - e\,\varphi(\mathbf{r}, t) + e\,\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t)$$

yields the correct equation of motion for a particle with electric charge e moving in an electromagnetic field, namely

$$m\ddot{\mathbf{r}} = e\,(\mathbf{E} + \dot{\mathbf{r}} \times \mathbf{B})$$

where

$$\mathbf{E} = -\nabla\varphi - \frac{\partial \mathbf{A}}{\partial t}$$
$$\mathbf{B} = \nabla \times \mathbf{A}$$

are the electric and magnetic fields, respectively. [The vector field $\mathbf{A}$ is called the *vector potential*, and the scalar field φ is called the *scalar potential*. Both $\mathbf{A}$ and φ may be functions of x, y, z and t.]

(b) Find the conjugate momentum $\mathbf{p}$ in terms of the positions and velocities. Is $\mathbf{p}$ the ordinary linear momentum?

(c) Find the Hamiltonian $H(\mathbf{r}, \mathbf{p}, t)$.

(d) Find Hamilton's equations of motion, and show that they are equivalent to Lagrange's equations of motion.

(e) Under what circumstances is H conserved?