MATHEMATICS 0054 (Analytical Dynamics)
YEAR 2018–2019, TERM 2

PROBLEM SET #5

This problem set is due at the beginning of the noon lecture on Monday 4 March.

Topics: Lagrangian approach to mechanics.

Reading:
- Gregory, Classical Mechanics, Chapter 12 (handout).
- Handout #11: The Lagrangian approach to mechanics.

1. [You have seen this problem before!]

A pendulum is constructed by attaching a mass \( m \) to an unstretchable string of length \( l \). The upper end of the string is connected to the uppermost point on a fixed vertical disk of radius \( R \), as shown in the diagram. Assume that \( l > (\pi/2)R \).

Obtain the Lagrangian of this system in terms of the generalized coordinate \( \varphi \), and derive the exact equation of motion. [You already did last week almost all the work needed for this.]

Remark: When we considered this problem before, we got the equation of motion from energy conservation. Note that in this case of a conservative system with one degree of freedom, the Lagrangian method is no simpler than the energy-conservation method: after all, if we can write \( L = T - V \), then we can also write \( E = T + V \)! The Lagrangian method becomes advantageous when dealing with constrained systems with \( n \geq 2 \) degrees of freedom: for these systems, energy conservation alone does not suffice to give the full equations of motion (we need \( n \) independent equations, but energy conservation only gives one).
2. A bead slides under the influence of gravity on the frictionless interior surface of the paraboloid of revolution $z = (x^2 + y^2)/2a = r^2/2a$. Use cylindrical coordinates $(r, \varphi, z)$.

Let us first analyze this problem by Newtonian methods:

(a) Write the $r$, $\varphi$ and $z$ components of Newton’s equations of motion for the bead. Your equations will of course contain an unknown constraint force.

(b) Find the equations of motion for the bead coordinates $(r, \varphi)$, by eliminating $z$ and the constraint force.

(c) Find two conserved quantities. [Hint: In what directions do the forces point? What, in addition to energy, will therefore be conserved?]

(d) Find a closed equation of motion for $r$ alone. One of the conserved quantities will appear as a parameter in your equation. [Hint: Recall the solution of the central-force problem.]

(e) Find the speed $v_0$ at which the bead will move in a horizontal circle of radius $r_0$.

(f) Find the frequency of small radial oscillations around the circular motion found in part (e).

Now let’s try it by Lagrangian methods:

(g) Find the Lagrangian and obtain the equations of motion for the bead coordinates $(r, \varphi)$. Does it agree with what you found previously by Newtonian methods?

(h) Show that the coordinate $\varphi$ is cyclic, and hence that the conjugate momentum $p_\varphi$ is conserved. Does this agree with what you found previously by Newtonian methods? What is the physical significance of $p_\varphi$?

3. A smooth thin wire is bent into the shape of a parabola, $z = x^2/2a$, and is made to rotate with a constant angular velocity $\omega$ about the $z$ axis [i.e. about the point $x = 0$ on the wire]; here the $+z$ direction is of course oriented upwards. A bead of mass $m$ then slides frictionlessly on the wire under the influence of gravity. Use cylindrical coordinates $(r, \varphi, z)$.

Let us first analyze this problem by Newtonian methods:

(a) Write the $r$, $\varphi$ and $z$ components of Newton’s equations of motion for the bead. Your equations will contain two unknown constraint forces.

(b) Use the equations of constraint to eliminate all reference to $\varphi$, $z$, and their time derivatives as well as to the constraint forces. That is, you should obtain a differential equation for $r$ alone.

(c) Show that the total mechanical energy $E$ of the bead is not conserved, and that the constraint force does work at a rate precisely $dE/dt$.

(d) Show that the equation of motion found in part (b) can be integrated once by the usual trick of multiplying it by $\dot{r}$. What is the relation between this result and part (c)?
(e) Integrate the equation of motion once more to get an “explicit” expression for $t$
as a function of $r$ (albeit in terms of an ugly integral).

Now let’s try it by Lagrangian methods:

(f) Write the Lagrangian in terms of the single degree of freedom $r$, and derive the
equation of motion. Does it agree with what you found previously by Newtonian
methods?

Note that this problem is a nontrivial test of the Lagrangian formalism, as it involves
a time-dependent constraint. In particular, the constraint force does work, so that
the total energy $E$ is not conserved. Nevertheless, the Lagrangian formalism gives the
correct equation of motion, without fuss.