

MATHEMATICS 0054 (Analytical Dynamics)
YEAR 2023–2024, TERM 2

PROBLEM SET #3

This problem set is due at the *beginning* of class on Thursday 8 February.

Topics: Coupled oscillations and normal modes. Standing waves on a linear chain. Introduction to waves.

Readings:

- Handout #8: Coupled oscillations and normal modes.
- Handout #9: Introduction to waves. [This is for “enrichment”, as I did not attempt to cover it in detail or to assign any problems on it.]

1. Consider a linear triatomic molecule (e.g. carbon dioxide), which we will model as a central particle of mass M connected on the left and right to particles of mass m , where each connection is via a spring of spring constant k and equilibrium length ℓ . Let us number these particles from left to right as 1,2,3, and let x_i be the displacement of particle # i from its equilibrium position. (We consider only one-dimensional motion along the given line.)

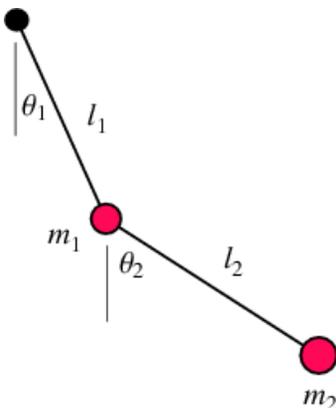
- (a) Find the equations of motion of this system.
- (b) Find the normal modes and, for each normal mode, describe the associated motion in words. What is the meaning of the normal mode with eigenfrequency 0? [*Hint:* When $\omega = 0$, we have $i\omega = -i\omega$, so there is another solution to the differential equation $\ddot{x} + \omega^2 x = 0$ besides $e^{\pm i\omega t}$. What is it?]

2. Consider a chain of n particles (each of mass m) joined by n springs (each of spring constant k and equilibrium length ℓ) as follows: particle #1 is connected to a fixed left wall by spring #1, particle #2 is connected to particle #1 by spring #2, and so forth, until particle # n is connected to particle # $n - 1$ by spring # n . So the situation is the same as that considered in Section 4 of Handout #8, *except that* the right wall and the rightmost spring are absent. Let x_i be the displacement of particle # i from its equilibrium position.

- (a) Find the equations of motion of this system.
- (b) For the case $n = 2$, find the normal modes.
- (c) *Optional:* Find the normal modes for general n . [*Hint:* Imitate the trick used in Section 4 of Handout #8, but with $f_{n+1} = f_n$ instead of $f_{n+1} = 0$.]

- (d) Now suppose that this chain is hanging vertically in the earth's gravitational field, rather than horizontally. Find the new equations of motion, and solve them. [*Hint*: Make the substitution $x_i = x'_i + c_i$; for suitably chosen constants c_i , you can reduce the new equations to those found in part (a). And even if you are unable to find the c_i , you should be able to explain the *qualitative* nature of the solution and how it relates to the one found in parts (a)–(c). Moreover, you should be able to find the c_i at least in the case $n = 2$.]

3. A double pendulum consists of one pendulum attached to another, as shown:



The two rods are massless and rigid.

- (a) Find the Cartesian coordinates (x_1, y_1) and (x_2, y_2) of the two masses in terms of θ_1 and θ_2 . Take the origin to be at the top support point. Make explicit what sign convention you are using for y . (And make sure you stick to it in the remainder of the problem!)
- (b) Compute \ddot{x}_1 , \ddot{y}_1 , \ddot{x}_2 and \ddot{y}_2 in terms of θ_1 , θ_2 and their time derivatives.
- (c) Find the forces acting on the two masses and, using your result from part (b), write the Newtonian equations of motion in terms of θ_1 and θ_2 . Your equations will involve two unknown tensions.
- (d) *Optional*: Eliminate the tensions to find a pair of coupled nonlinear differential equations for θ_1 and θ_2 . [*Hint*: Work first on the \ddot{x}_2 and \ddot{y}_2 equations and form a linear combination of them to eliminate T_2 . Then work on the equations for $m_1\ddot{x}_1 + m_2\ddot{x}_2$ and $m_1\ddot{y}_1 + m_2\ddot{y}_2$ (these combinations already eliminate T_2 — why?) and form a linear combination of them to eliminate T_1 .] Show that these equations are

$$\begin{aligned} (m_1 + m_2)l_1\ddot{\theta}_1 + m_2l_2\ddot{\theta}_2 \cos(\theta_1 - \theta_2) + m_2l_2\dot{\theta}_2^2 \sin(\theta_1 - \theta_2) + (m_1 + m_2)g \sin \theta_1 &= 0 \\ m_2l_2\ddot{\theta}_2 + m_2l_1\ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2l_1\dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + m_2g \sin \theta_2 &= 0 \end{aligned}$$

[I am giving you these equations so that you'll be able to do parts (e) and (f) even if you had trouble doing parts (b)–(d).] Do these equations make sense in the limit $m_2/m_1 \rightarrow 0$?

- (e) Find the linearized equations when θ_1 and θ_2 are assumed small.
- (f) For the case $l_1 = l_2 = l$ (but m_1 and m_2 arbitrary), find the normal modes.

[**Remark.** The double pendulum is especially interesting when one does *not* make the small-angle approximation, but instead studies the full nonlinear equations. Then the motion can be *chaotic*, i.e. exhibit sensitive dependence to initial conditions. Computer simulations of the double pendulum can be found at numerous places on the web. Stunning videos of real double pendulums can be found at <http://www.youtube.com/watch?v=z3W5aw-VKKA> and <http://www.youtube.com/watch?v=U39RMUzCjiU> Note that, in these videos, friction causes the amplitude to very gradually decrease, so that one eventually leaves the chaotic regime and enters the small-oscillations regime.]