MATHEMATICS 0054 (Analytical Dynamics)
YEAR 2021–2022, TERM 2

PROBLEM SET #2

This problem set is due at the *beginning* of the *noon* class on Monday 31 January.

Topics:

- Solvable cases of one-dimensional motion.
- Systems of particles and conservation laws: linear momentum, angular momentum, energy (internal and external potentials).
- Coupled oscillations and normal modes. Standing waves on a linear chain. [Next week I will give you problems on coupled oscillations.]

Readings:

- Handout #5: Solvable cases of one-dimensional motion.
- Handout #6: Momentum, angular momentum, and energy; conservation laws.
- Handout #8: Coupled oscillations and normal modes.

Warning: This problem set is rather long, and the problems are challenging; I urge you to get started *early* and give yourself enough time!

1. Consider a particle with initial velocity $v_0 > 0$, subject only to the retarding force $F = -kv|v|^{n-1}$ with $k, n > 0$. Find $v(t)$ and $x(t)$, and investigate the behavior of v and x as $t \to +\infty$. It will turn out (but you have to prove this!) that there are three cases:

   (a) For a certain interval of *small* n, the particle comes to rest after a finite time, and thus has travelled a finite distance.

   (b) For a certain interval of *intermediate* n, the particle comes to rest only asymptotically as $t \to +\infty$, but the distance it travels as $t \to +\infty$ is finite.

   (c) For a certain interval of *large* n, the particle comes to rest only asymptotically as $t \to +\infty$, and it travels an infinite distance as $t \to +\infty$.

   Prove this scenario, find the values of n that form the dividing lines between these three cases, and put the dividing-line values of n into the correct cases.

2. An open railway car of mass M rolls frictionlessly on a horizontal track, and is acted upon by a constant horizontal force F_0. At $t = 0$ the car has velocity v_0, and rain begins to fall vertically with respect to the ground. Rainwater enters the car at a constant rate α (mass/time) and leaks out through a small hole in the bottom of the car at a constant rate β, with $\alpha > \beta$.

   (a) Find the equation of motion of the car — that is, find a differential equation for the car's velocity $v(t)$. [*Hint:* Look at the *same* collection of particles (the "system") at two nearby times, t and $t + \Delta t$, and write that the rate of change of the system's total momentum equals the total external force on the system.] [*Remark:* Note that this equation does *not* depend on α and β only through the combination $\alpha - \beta$, as one might naively expect. That is because there is a real physical difference between the water entering the car and the water leaking out of the car — can you see what this difference is?]

   (b) Solve this equation to find the velocity as a function of time.

3. A uniform heavy chain of length a is partly sitting on a table, partly hanging down over the edge. Initially a part of length b $(< a)$ hangs over the edge (with zero initial velocity), while the remaining part of length $a - b$ is coiled up at the edge of the table.

   (a) Find the equation of motion for the amount $x(t)$ hanging over the edge.

   (b) Solve this equation to find the chain's speed v as a function of x, and in particular to find the chain's speed v_{final} when the last link leaves the edge of the table. [*Hint:* $\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt}$.]

   (c) Use your solution to compute the kinetic and potential energies as a function of x. Is the total energy (kinetic + potential) conserved, gained or lost? Explain physically. [*Remark:* The physical explanation is a bit subtle.]

   Same problem, but this time the part of the chain sitting on the table is stretched out to its full length in the direction perpendicular to the edge of the table. The table is frictionless.

   (d) Find the equation of motion for the amount $x(t)$ hanging over the edge.

   (e) Solve this equation to find the chain's speed v as a function of x, and in particular to find the chain's speed v_{final} when the last link leaves the edge of the table.

   (f) Use your solution to compute the kinetic and potential energies as a function of x. Is the total energy (kinetic + potential) conserved, gained or lost? Explain physically.

4. And now for the famous problem of a raindrop falling through mist, collecting mass as it falls. Suppose that at time t the raindrop has a mass $m(t)$ and a downward velocity $v(t)$. First we need to find a pair of coupled differential equations for the two unknown functions $m(t)$ and $v(t)$; then we need to solve them.

   The first differential equation is the Newtonian equation of motion for the raindrop:

(a) Derive the Newtonian equation of motion by looking at the *same* collection of water particles (the "system") at two nearby times, t and $t + \Delta t$, and writing that the rate of change of the system's total momentum equals the total external force on the system.

The second differential equation states the hypothesized law of accretion for the raindrop, and we will consider two versions:

(b) Assume that the raindrop remains spherical and that the rate of accretion of mass is proportional to the drop's surface area. Write the equation for dm/dt that this implies. Then solve the system of two coupled differential equations, assuming that the drop starts from rest when it is infinitely small; show that its acceleration is constant and is equal to $g/4$.

(c) Assume that the raindrop remains spherical and that the rate of accretion of mass is proportional to the volume swept out as it falls (i.e. is proportional to the drop's cross-sectional area multiplied by its speed of fall). Write the equation for dm/dt that this implies. Then solve the system of two coupled differential equations, assuming that the drop starts from rest when it is infinitely small; show that its acceleration is constant and is equal to $g/7$.

[*Comment:* (b) is fairly straightforward, because the rate of accretion dm/dt depends only on m, not on v; therefore you can first solve the accretion equation to find $m(t)$, then plug this in to the Newtonian equation of motion to find $v(t)$. But (c) is really tricky, because the two equations are coupled! *Hint for decoupling them:* Temporarily consider m to be the independent variable instead of t; find and solve a differential equation for the unknown function $v(m)$. Then use this to find $v(t)$.]