Random and Chaotic Oscillations in a Model of Childhood Epidemics Caused by Seasonal Variations of the Contact Rate

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Abstract. We consider an alternative approach to a standard model for seasonal oscillations of childhood infections by introducing a noisy variation of control parameter instead of periodic one. Chaotic and noise-induced oscillations are compared and the problem of distinguishing between these two kinds of oscillations is addressed. In a certain range of the action frequencies the synchronization of noise-induced oscillations takes place in the sense that the mean frequency of the oscillations becomes close to the action frequency. We demonstrate also the effect of synchronization between two interacting populations.

INTRODUCTION

The nature of irregularity is a challenging problem in ecology [1], and in epidemiology of host populations in particular. An epidemiological, or so-called SEIR model for seasonal oscillations of childhood infections, such as chickenpox, measles, mumps and rubella, under the influence of contact rate fluctuations, was suggested by Dietz [2]. Later this model was intensively

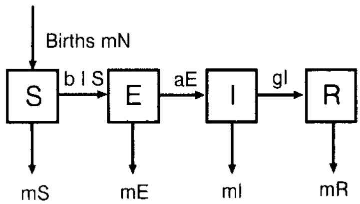

FIGURE 1. The scheme of the compartmental SEIR model.

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studied (see, e.g. [3,4]) and it was shown that periodic variations of the contact rate result in chaotic oscillations of childhood infections. The model involves four components: (1) Susceptibles (S); (2) Exposed but not yet infective (E); (3) Infective (I); (4) Recovered and immune (R). Mutual relations between these components are illustrated schematically in Fig. 1.

The model has the following form:

$$
\begin{aligned}
\dot{S} &= m(1 - S) - b(t)SI, \\
\dot{E} &= b(t)SI - (m + a)E, \qquad (1) \\
\dot{I} &= aE - (m + g)I, \\
\dot{R} &= gI - mR, \qquad (2)
\end{aligned}
$$

where $1/m$ is the average expectancy time, $1/a$ is the average latency period, $1/g$ is the average infection period, b is the contact rate (the average number of susceptibles contacted yearly with infective). The total number of children N is normalized to 1. The assumptions of the derivation can be found in [4].

In experimental data the strong seasonality in outbreaks of epidemics has been found. London and Yorke [5] addressed the problem, whether there is an underlying seasonal variation in the contact rate. Usually the seasonal variations are taken into account in the form: $b(t) = b_0(1 + b_1 f(t))$, where $f(t)$ is a function describing the shape of the contact rate variation. If the parameter b varies with time then the variables S, E and I oscillate around a stable singular point with coordinates ($S_0 = (m + a)(m + g)/ab_0$, $E_0 = m/(m + a) - m(m + g)/ab_0$, $I_0 = am/((m + a)(m + g)) - m/b_0$).

PERIODIC AND RANDOM VARIATION OF THE CONTACT RATE

In [3] it was assumed that owing to seasonal variations of environmental conditions the contact rate b depends periodically on time with the period equal to one year, *viz.*, $f(t) = \cos(2\pi t)$ (time is measured in years).

In this case either periodic or chaotic oscillations of S, E, and I appear. The transition from periodic to chaotic oscillations as the parameter b_1 increases occurs via the sequence of period-doubling bifurcations. Chaotic oscillations were observed by Olsen and Schaffer [3] for the following values of the parameters: $m = 0.02\,\text{year}^{-1}$, $a = 35.84\,\text{year}^{-1}$, $g = 100\,\text{year}^{-1}$, $b_0 = 1800\,\text{year}^{-1}$, $b_1 = 0.28$ (Fig. 2 *a*). These parameters correspond to estimates made for childhood diseases in first world countries.

From a physical standpoint, random variation of the contact rate is more justified than periodic. For the case when $f(t) = \xi(t)$ is a colored (band limited white) noise with the center frequency $\omega = 2\pi$, the results of numerical simulation of Eqs. (1) with the integration step 10^{-4} are shown in Fig. 2*a* for the same values of the parameters as for periodic variation of contact rate (b_1

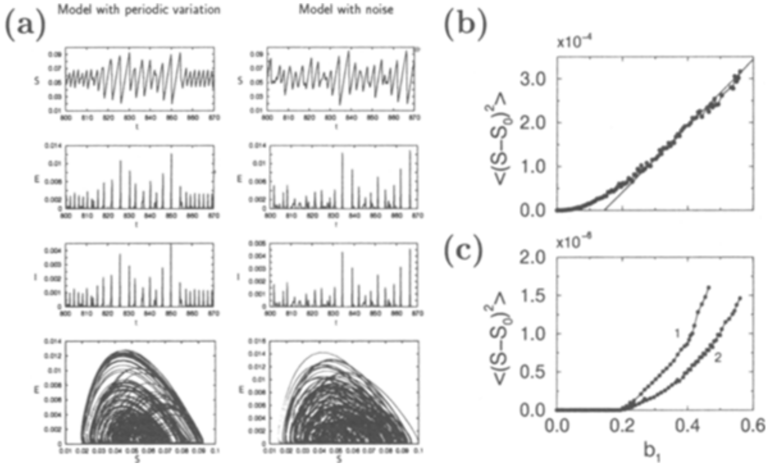

FIGURE 2. *(a)* Chaotic ($f(t) = \cos(2\pi t)$) and noise-induced ($f(t) = \xi(t)$) oscillations. Solutions and their projection on the (E, S) plane. *(b)* the dependence of the variance of S on the parameter b_1, *(c)* the same dependence, if the additive noise is absent. The term $-\alpha E'^3$ is added to the equation for E' to avoid instability. (*curve 1*: $\alpha = 20 \cdot 10^6$, *curve 2*: $\alpha = 30 \cdot 10^6$).

is chosen so that the variance of $S(t)$ would be approximately the same as for $f(t) = \cos(2\pi t)$. As the parameter b_1 increases the variance of $S(t) - S_0$ increases too (see Fig. 2b).

In a certain range of b_1 the dependence of $\overline{(S - S_0)^2}$ on b_1 can be approximated by a straight line. The value of b_1 for which this straight line intersects the abscissa's axis can be taken for the point of the noise-induced phase transition (in the sense, discussed in [6]). It seems reasonable to say that this phase transition differs in the mechanism of its appearance from that considered in [6] for a pendulum with a randomly vibrating suspension axis. The difference lies in the fact that this phase transition is caused primarily by additive noise but not multiplicative. To prove this let us rewrite model equations (1) for new variables $S' = S - S_0$, $E' = E - E_0$, $I' = I - I_0$ to detach additive noise:

$$\begin{aligned}
\dot{S}' &= -mS' - b(S'I' + S'I_0 + S_0I') - b_0b_1S_0I_0\xi(t),\\
\dot{E}' &= b(S'I' + S'I_0 + S_0I') - (m+a)E' + b_0b_1S_0I_0\xi(t),\\
\dot{I}' &= aE' - (m+g)I'
\end{aligned} \tag{3}$$

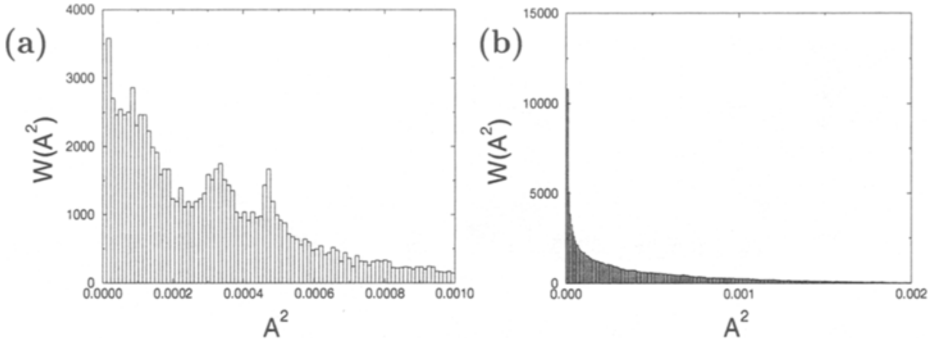

FIGURE 3. The histogram for the instantaneous amplitude squared in the case of periodic (a) and random (b) variation of the contact rate.

If in Eqs. (3) we artificially eliminate additive noise (the term $b_0 b_1 S_0 I_0 \xi(t)$), we obtain that the threshold value of b_1 for which the phase transition appears is larger than for the original equations (1)(see. Fig. 3b).

Comparing noise-induced and chaotic oscillations shown in Figs. 2a we can conclude that the oscillations are at least superficially similar.

DISTINGUISHING CHAOTIC AND NOISE-INDUCED OSCILLATIONS

Despite the similarity, distinctions between these two kinds of oscillations exist. First and foremost the power spectra of the oscillations excited are different. In the case of harmonic variation of the contact rate the power spectrum contains a discrete line at the frequency of this variation. In addition, in the spectrum there are peaks at subharmonics of the variation frequency. In the case of random vibration of the contact rate the power spectrum contains no discrete lines. The second essential difference manifests itself in the correlation dimension of the attractor reconstructed from time series. In distinction to noise-induced oscillations the correlation dimension estimated in simulations for the harmonic variation of the contact rate saturates as the embedding space dimension increases.

In the case of random variation of the contact rate the calculated correlation dimension increase monotonically with the increase in the embedding space dimension. The latter indicates that the correlation dimension of the attractor are infinite or very large. Such a dissimilarity of the result obtained for noise-induced pendulum's oscillations [6], for which the correlation dimension was found to be finite, is attributable to the influence of additive noise.

Finally, the noise-induced and chaotic oscillations can be distinguished by use of the Rytov-Dimentberg criterion, initially proposed in [8,9] to solve the problem of distinguishing between noise passed through a linear narrow-band

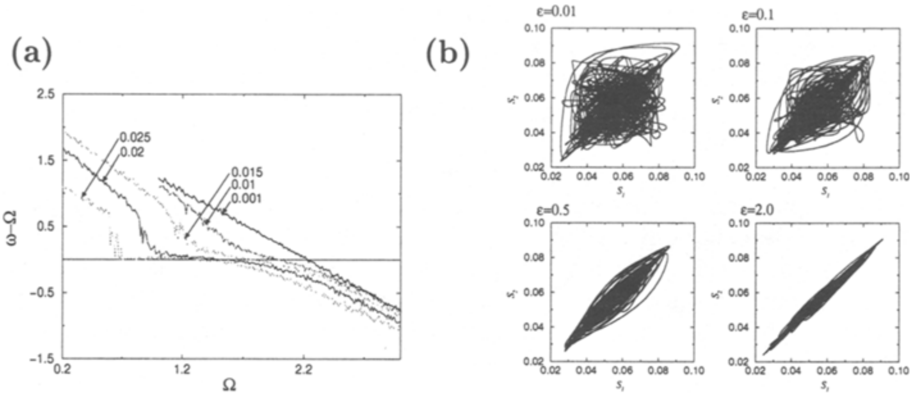

FIGURE 4. (a) The dependence of the difference $\omega - \Omega$ on the frequency of the external force Ω for different values of ε (shown in the fig.). (b) The illustration of the synchronization between two identical systems in the phase space $(S_1; S_2)$. The different amplitude of the coupling is shown in the figure.

filter and periodic but noisy self-oscillations. According to this criterion, the probability distributions for the process itself and for the instantaneous amplitude squared are monotonic in the case of noise-induced oscillations, whereas for chaotic oscillations these distributions have to have peaks. The distinction in the probability distribution for the instantaneous amplitude squared is illustrated in Fig. 3 *a,b*. The instantaneous amplitude was calculated by means of the Hilbert transform (see [7]).

SYNCHRONIZATION OF NOISE-INDUCED OSCILLATIONS

Another interesting question is the interaction of two neighbouring populations, for example the synchronization. To study it we first consider the simplest case of the synchronization by the external force. We add the term $\varepsilon\cos(\Omega t)$ to r.h.s of the eq. (1) for S , which can model, for example, the periodic influence of economical cycles on the number of susceptible children. Here ε, Ω are the amplitude and frequency of the external force. We have found that in a certain range of the force frequencies, a synchronization of oscillations takes place in the sense that the mean frequency of noise-induced oscillations ω is approximately equal to the action frequency, if ε is large enough (Fig. 4 *a*). Such interpretation of the problem of the synchronization and calculation technique was proposed in [7,10] and used in [11].

It is interesting to note that frequency locking can be also treated as the possibility to control the frequency of oscillations by the harmonic action.

Another kind of synchronization is the mutual synchronization between two populations, interacted, for example, by travel of groups of possibly infected children in remote areas. To study it we consider two populations, described by eqs. (1) each, coupled by the term $\varepsilon(S_{2,1} - S_{1,2})$, added to the equation for S. Indexes $1, 2$ correspond to different subpopulations, and the amplitude of coupling is given by ε.

The increase of coupling strength leads to the almost full synchronization of two populations. This can be illustrated by Fig. 4*b*, where the solution of coupled model equations is shown for different values of ε. For large enough ε the projection of the solution to the plane (S_1, S_2) is almost a straight line, what corresponds to the case of the synchronization between two populations. We note that despite the synchronization, solution of the model remains random.

SUMMARY

To summarize, we have studied the behaviour of the model with random variation of the contact rate and compared chaotic and noise-induced oscillations. We have shown that the model undergoes phase transition (in the sense discussed in [6]). The Rytov-Dimentberg criterion is found to be valid for distinguishing chaotic and noise-induced oscillations in the model considered. In a certain range of the action frequencies, a synchronization of noise-induced oscillations by the external force takes place. We demonstrate also the effect of the almost full synchronization between noise-induced oscillations in two populations.

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