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A Nonlinear Model of the Helmholtz Resonator with a Movable Wall

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Abstract—A nonlinear model of the Helmholtz resonator with a movable wall is developed. The energy reflectance, transmittance, and absorbance, as well as the energy distribution, are calculated as functions of frequency with allowance for Stokes friction and the friction of an acoustic boundary layer. Characteristics necessary for such a system to be used as a sound absorber or sound insulator are considered.

At high levels of sound pressure, the response of a concentrated oscillatory system—a Helmholtz resonator (HR)—to an external stimulus in the form of an acoustic wave is known to become nonlinear [1]. This nonlinearity gives rise to harmonics, combination frequencies [2] and amplitude-dependent shifts of the maximum of the HR frequency response [3]. Most treatises on resonance sound absorbers assume a fixed back wall of the resonator. Rudenko and Khizhnykh [4] have proposed a nonlinear model of the HR that describes all said nonlinear phenomena in systems where HRs may be used to absorb intense acoustic or shock waves.

Rzhevkin *et al.* [5, 6] have studied resonators with a compliant front wall and the radiation of a panel coated with sound absorber. In real situations with an oscillating back wall of the resonator, one have to cope not only with a reflected wave but also with a transmitted wave. This model not only alters the characteristics of the HR as a sound absorbing system, but also compels one to consider, unlike [4], a concomitant sound insulation problem.

In this paper, we extend the model of Rudenko and Khizhnykh [4] sketched in Fig. 1. We consider an oscillatory system consisting of a throat (*1*) made as a cylindrical tube of radius a, and a cavity (2) with a movable back wall (*3*). The throat is filled with a viscous incompressible liquid of density ρ_l, the cavity is filled with air of density ρ_0. The system is irradiated by a sound wave incident from the left. This wave sets in motion the liquid in the throat and, via the compressible air in the cavity, the back wall of the HR. A part of energy of the incident wave is absorbed due to the viscosity of the throat liquid, the friction of the oscillating wall, and reradiation into the reflected and transmitted waves.

Let us construct a mathematical model that describes the oscillations of the wall and the throat liquid. The equation of motion for the wall is

$$m\frac{d^2z}{dt^2} = -kz + p(x = L, t)S' - p_{ac2}(z = 0, t)S', \quad (1)$$

where z is the travel of the wall from the equilibrium position, m is the mass of the wall, and p and p_{ac2} are the pressure at the input into the cavity and the acoustic pressure behind the wall. We assume that the oscillations of the back wall are bending plate oscillations, and we will consider only their "piston" mode.

The pressure at the resonator input may be defined using the adiabaticity of forcing a liquid into the cavity:

$$p(x = L, t) = -c_0^2\rho\frac{V'}{V_0} = \frac{c_0^2\rho_0}{V_0} \times\int_0^a \xi(x = L, r, t)2\pi r dr - \frac{c_0^2\rho S'}{V_0}z, \quad (2)$$

where V' is the increment of the volume in the cavity, and ξ is the displacement of liquid in the throat.

Using (2) and differentiating (1) with respect to time, we obtain the equation for the wall velocity

$$\frac{d^2u}{dt^2} + \left[\frac{k}{m} + \frac{c_0^2\rho_0 S'^2}{V_0 m}\right]u = \frac{c_0^2\rho}{V_0 m} \times\int_0^a V_x(x = L, r, t)2\pi r dr - \frac{S'}{m}\frac{dp_{ac2}}{dt}(z = 0, t). \quad (3)$$

In order to describe the motion of liquid in the throat, we use simplified equations of quasi-uniform flow [7, 8]

$$\frac{\partial V_x}{\partial t} + V_x\frac{\partial V_x}{\partial x} + V_r\frac{\partial V_x}{\partial r} - \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_x}{\partial r}\right) = -\frac{1}{\rho_l}\frac{\partial}{\partial x}p(x, t), \quad (4)$$

$$\frac{\partial V_x}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}(rV_r) = 0, \quad (5)$$

where V_x and V_r are the longitudinal and transverse velocity components of the liquid and ν is the shear viscosity.

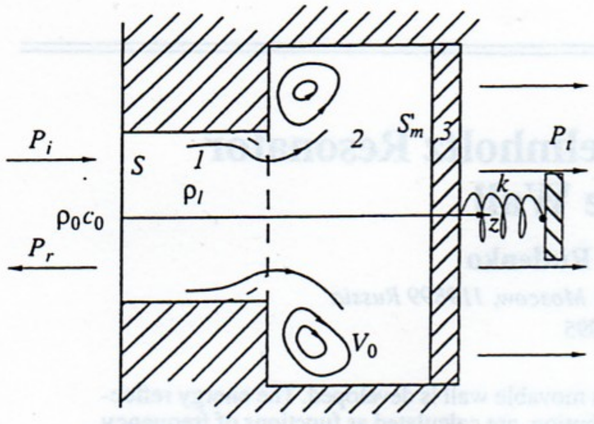

Fig. 1. Helmholtz resonator with a movable wall.

These equations suggest that the pressure must be a linear function of x: $p(x, y) = p_{ac2}(t) + x\beta(t)$ [4]; at $x = 0$ it must be equal to the pressure at the input to the throat $p_{ac}(x = 0, t)$, and at $x = L$, it must be $p(x = L, t)$. Now, we easily obtain the right side of equation (4). We construct an approximate solution of the equations for flows in the throat by introducing the longitudinal velocities averaged over the cross section and over the volume of the throat:

$$\overline{V} = \frac{1}{S}\int_0^a V_x 2\pi r dr, \quad \overline{\overline{V}} = \frac{1}{SL}\int_0^L dx \int_0^a V_x 2\pi r dr, \qquad (6)$$

where $S = \pi a^2$. We note that, from the continuity, it follows that $\overline{V} = \overline{V}(t)$; a flow over the cross-section does not depend on the coordinate x. This conclusion agrees with the fact that we seek a solution in the low-frequency region.

Combining equations (3)–(6) and averaging over the volume, we obtain

$$\frac{d^2\overline{\overline{V}}}{dt^2} + \frac{c_0^2\rho_0}{V_0\rho_l L}\overline{\overline{V}} + D_1(\overline{\overline{V}}) + D_2(\overline{\overline{V}}) + \frac{1}{L}\frac{d}{dt}$$
$$\times[\overline{V}^2(L, t) - \overline{V}^2(0, t)] = \frac{1}{\rho_l L}\frac{dp_{ac}(t)}{dt} + \frac{c_0\rho_0 S'}{\rho_l L V_0}u, \qquad (7)$$

where [7]

$$D_1(\overline{\overline{V}}) = \frac{1}{\pi a}\sqrt{\frac{\nu}{\pi}}\frac{d}{dt}\int_{-\infty}^{t}\frac{d\overline{\overline{V}}}{dt'}\frac{dt'}{\sqrt{t-t'}} \qquad (8)$$

is the dissipative term responsible for the acoustic boundary layer, and

$$D_2(\overline{\overline{V}}) = \nu a_2 n\frac{d\overline{\overline{V}}}{dt} \qquad (9)$$

is a second dissipative term responsible for the Stokes friction.

If the throat is filled with a material that has a surface considerably exceeding that of the pipe, then πa in (8) should be replaced with a constant of length dimension: $a_1 = a/N$, where N takes into account the increase of the surface of the boundary layer, a_2 is a second empirical constant with a magnitude of the order of the typical dimension of elements passed by the stream, and n is the volume concentration of these elements (the physical meaning of these constants is discussed in depth in [4, 9, 10]). For simplicity, we further omit the overbars and write v as V.

The nonlinear term in equation (7) may be approximated by a power expansion [4, 9, 10]

$$I = (\gamma_1 V + \gamma_2 V^2)dV/dt. \qquad (10)$$

Equations (3) and (7) describe two oscillatory circuits: the throat and wall; however, for a complete problem statement, we need to introduce boundary conditions that will match the hydrodynamic and acoustic parts of the problem. For $x < 0$, the acoustic field is constituted by the incident and reflected waves:

$$p_{ac}(x, t) = p_i(t - x/c_0) + p_r(t + x/c_0),$$
$$u_{ac}(x, t) = \frac{1}{\rho_0 c_0}[p_i(t - x/c_0) - p_r(t + x/c_0)], \qquad (11)$$

where u_{ac} is the oscillatory velocity.

Now, with the matching conditions:

$$u_{ac}(0, t) = V(0, t), p_{ac2}(z = 0, t) = \rho_0 c_0 u,$$

the equations of system motion take the form

$$\frac{d^2u}{dt^2} + 2\delta_1\frac{du}{dt} + \omega_{01}^2 u = \kappa_1 V, \qquad (12)$$

$$\frac{d^2V}{dt^2} + 2\delta_2\frac{dV}{dt} + \omega_{02}^2 V + \frac{1}{a_1}\sqrt{\frac{\nu}{\pi}}\frac{d}{dt}\int_{-\infty}^{t}\frac{dV}{dt'}\frac{dt'}{\sqrt{t-t'}}$$
$$+ (\gamma_1 V + \gamma_2 V^2)\frac{dV}{dt} = \frac{2}{\rho_l L}\frac{dp_i(t)}{dt} + \kappa_2 u, \qquad (13)$$

where

$$\omega_{01}^2 = k/m + c_0^2\rho_0 S'^2/V_0 m \text{ and } \omega_{02}^2 = c_0^2\rho_0 S/\rho_l V_0 L$$

are the eigenfrequencies of the wall and throat in the absence of nonlinearity and friction,

$$\delta_1 = \rho_0 c_0 S'/2m \text{ and } \delta_2 = \nu a_2 n/2 + \rho_0 c_0/2\rho_l L$$

are the losses due to friction and radiation (the wall friction may be taken into account by adding a friction factor to δ_1), and

$$\kappa_1 = c_0^2\rho_0 SS'/mV_0 \text{ and } \kappa_2 = c_0^2\rho_0 S'/\rho_l V_0 L$$

are the coefficients of coupling between two oscillatory circuits.