System Size Stochastic and Coherence Resonance

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Abstract. We demonstrate that in ensembles of noise-driven dynamical systems a maximum order can appear at a certain system size, while the noise and coupling remain constant. We discuss two situations with system size resonance. In the first one an ensemble of coupled nonlinear noisy oscillators demonstrates in the thermodynamic limit an Ising-type transition. When a small periodic force acts on the ensemble in the ordered phase, the linear response of the system has a maximum at a certain system size. Next, we show that in an ensemble of coupled noise-driven FitzHigh-Nagumo oscillators a maximum coherence of the output mean field occurs at a definite ensemble size.

INTRODUCTION

Constructive role of noise in nonlinear systems has been attracted large interest recently. One extremely popular example is ***Stochastic Resonance*** [1, 2]. As was demonstrated in [3], a response of a noisy nonlinear system to a periodic forcing can exhibit a resonance-like dependence on the noise intensity. In other words, there exists a "resonant" noise intensity at which the response to a periodic force is maximally ordered. Stochastic resonance has been observed in numerous experiments.

Being first discussed in the context of a simple bistable model, stochastic resonance has been also studied in complex systems consisting of many elementary bistable cells [4]. Again, one observes a resonance-like dependence on the noise intensity, moreover, the resonance may be enhanced due to coupling [5, 6]. Here in the next section we discuss another type of resonance in such systems, namely the ***System Size Stochastic Resonance***, when the dynamics is maximally ordered at a certain number of interacting subsystems [7]. Contrary to previous reports of array-enhanced stochastic resonance phenomena (cf. also [8, 9]), here we fix the noise strength, coupling, and other parameters; only the size of the ensemble changes.

Noteworthy, the order in a noise-driven system can have a maximum at a certain noise level even in the absence of periodic forcing, this phenomenon being called ***Coherence Resonance*** [10, 11, 12]. Below we demonstrate, taking an ensemble of coupled noise-driven FitzHugh–Nagumo systems, that it also can appear in the form of ***System Size Coherence Resonance***.

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SYSTEM SIZE STOCHASTIC RESONANCE

The basic model to be considered below is the ensemble of noise-driven bistable overdamped oscillators, governed by the Langevin equations

$$\dot{x}_i = x_i - x_i^3 + \frac{g}{N}\sum_{j=1}^{N}(x_j - x_i) + \sqrt{2D}\xi_i(t) + f(t)\,. \tag{1}$$

Here $\xi_i(t)$ is a Gaussian white noise with zero mean: $\langle \xi_i(t)\xi_j(t')\rangle = \delta_{ij}\delta(t-t')$; g is the coupling constant; N is the number of elements in the ensemble, and $f(t)$ is a periodic force to be specified later. In the absence of periodic force the model (1) has been extensively studied in the thermodynamic limit $N \to \infty$. It demonstrates an Ising-type phase transition at $g = g_c$ from the disordered state with vanishing mean field $X = N^{-1}\sum_i x_i$ to the "ferromagnetic" state with a nonzero mean field $X = \pm X_0$. A theory of this transition, based on the nonlinear Fokker-Planck equation, was developed in [13] where also the expressions for the critical coupling g_c are given.

While in the thermodynamic limit the full description of the dynamics is possible, for finite system sizes we have mainly a qualitative picture: in the ordered phase the mean field X switches between the values $\pm X_0$ and its average vanishes for all couplings. The rate of switchings depends on the system size and tends to zero as $N \to \infty$. The asymptotic dynamics in this limit has been discussed in [14].

For us, of the main importance is the fact that qualitatively the behavior of the mean field can be represented as the noise-induced dynamics in a potential with one minimum in the disordered phase (at $X = 0$) and two symmetric minima (at $X = \pm X_0$) in the ordered phase. Now applying the ideas of the stochastic resonance, one can expect in the bistable case (i.e. in the ordered phase for small enough noise or for large enough coupling) a resonant-like behavior of the response to a periodic external force when the intensity of the effective noise is changed. Because this intensity is inverse proportional to N, we obtain the resonance-like curve of the response in dependence of the system size. The main idea behind the system size resonance is that in finite ensembles of noise-driven or chaotic systems the dynamics of the mean field can be represented as driven by the effective noise whose variance is inverse proportional to the system size [14, 15, 16]. This idea has been applied to description of a transition to collective behavior in [17]. In [18] it was demonstrated that the finite-size fluctuations can cause a transition that disappears in the thermodynamic limit. The description of finite-size effects in deterministic chaotic systems using the effective noise concept has been suggested in [15]. We emphasize that noise plays an essential role in this picture: with $D = 0$ (1) is a deterministic oscillator (double or single well, depending on g), whose response to a periodic force does not depend on N.

Before proceeding to a quantitative analytic description of the phenomenon, we illustrate it with direct numerical simulations of the model (1), with a sinusoidal forcing term $f(t) = A\cos(\Omega t)$. Figure 1 shows the linear response function, i.e. the ratio of the spectral component in the mean field at frequency Ω and the amplitude of forcing A, in the limit $A \to 0$. For a given frequency Ω the dependence on the system size is a bell-shaped curve, with a pronounced maximum. The dynamics of the mean field $X(t)$ is illustrated in Fig. 2, for three different system sizes and for a particular frequency. The resonant dynamics (Fig. 2b) demonstrates a typical for stochastic resonance synchrony between the driving periodic force and the switchings of the field between the two stable positions. For non-resonant conditions (Fig. 2a,c) the switchings are either too frequent or too rare, as a result the response is small.

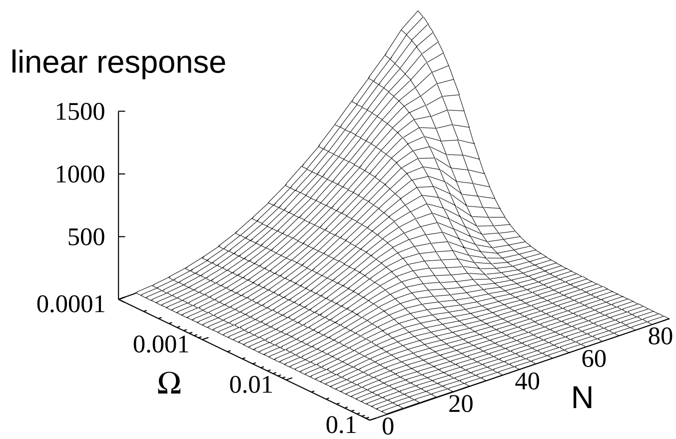

FIGURE 1. Linear response of the ensemble (1) ($D = 0.5$, $\varepsilon = 2$) in dependence on the frequency Ω and the system size N. Reprinted from [7].

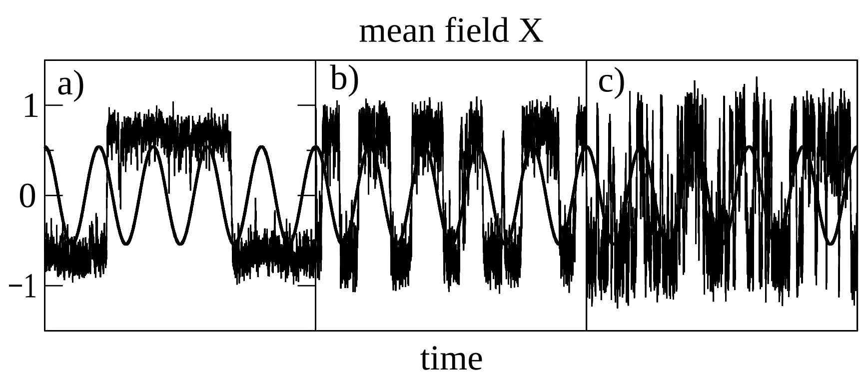

FIGURE 2. The time dependence of the mean field in the ensemble (1) for $D = 0.5$, $g = 2$, $A = 0.02$, $\Omega = \pi/300$, and different sizes of the ensemble: (a) $N = 80$, (b) $N = 35$, and (c) $N = 15$. We also depict the periodic force (its amplitude is not in scale) to demonstrate the synchrony of the switchings with the forcing in (b).

To describe the system size resonance analytically, we use, following [13], the Gaussian approximation. In this approximation one writes $x_i = X + \delta_i$ and assumes that δ_i are independent Gaussian random variables with zero mean and the variance M. Assuming furthermore that $N^{-1}\sum_i \delta_i^2 = M$ and neglecting the odd moments $N^{-1}\sum_i \delta_i$, $N^{-1}\sum_i \delta_i^3$, as well as the correlations between δ_i and δ_j, we obtain from (1) the equations for X and M:

$$\dot{X} = X - X^3 - 3MX + \sqrt{\frac{2D}{N}}\eta(t) + f(t)\,, \tag{2}$$

$$\frac{1}{2}\dot{M} = M - 3X^2M - 3M^2 - gM + D\,, \tag{3}$$

where η is the Gaussian white noise having the same properties as $\xi_i(t)$. In the thermodynamic limit $N \to \infty$ the noisy term η vanishes. If the forcing term is absent ($f = 0$), the equations coincide with those derived in [13]. This system of coupled nonlinear equations exhibits a pitchfork bifurcation of the equilibrium $X = 0,\ M > 0$ at $g_c = 3D$. This bifurcation is supercritical for $D > 2/3$ in accordance with the exact solution of (1) given in [13], below we consider only this case. For $g > g_c$ the system is bistable with two symmetric stable fixed points

$$X_0^2 = (2 - g + S)/4\,, \quad M_0 = (2 + g - S)/12\,, \tag{4}$$

(here $S = \sqrt{(2+g)^2 - 24D}$) and the unstable point $X = 0$, $M = (1 - g + \sqrt{(1-g)^2 + 12D})/6$. Now, with the external noise η and with the periodic force $f(t)$ the problem reduces to a standard problem in the theory of stochastic resonance, i.e. to the problem of the response of a noise-driven nonlinear bistable system to an external periodic force (because the noise affects only the variable X, it does not lead to unphysical negative values of variance M, since $\dot{M}$ is strictly positive at $M = 0$). This response has a maximum at a certain noise intensity, which according to (2) is directly related to the system size.

To obtain an analytical formula, we perform further simplification of the system (2),(3). Near the bifurcation point we can use the slaving principle to the dynamics of X is slower than that of M, and we can exclude the latter one assuming $\dot{M} \approx 0$. Then from (3) we can express M as a function of X and substitute to (2). Near the bifurcation point we obtain a standard noise-driven bistable system

$$\dot{X} = aX - bX^3 + \sqrt{\frac{2D}{N}}\eta(t) + f(t)\,, \tag{5}$$

where $a = 1 + 0.5(g-1) - 0.5\sqrt{(g-1)^2 + 12D}$, $b = -0.5 + 1.5(g-1)((g-1)^2 + 12D)^{-1/2}$. A better approximation valid also beyond a vicinity of the critical point can be constructed if we use $\bar{b} = aX_0^{-2}$ instead of b, where the fixed point X_0 is given by (4). Having written the ensemble dynamics as a standard noise-driven double-well system (5) (cf. [1, 2, 19]), we can use the analytic formula for the linear response R derived in [19]. It reads

$$R = \frac{NX_0^2}{2Da}\left(\frac{\mathscr{D}_{-3/2}(-\sqrt{s})}{\mathscr{D}_{-1/2}(-\sqrt{s})}\right)^2 \left[1 + \frac{\pi^2\Omega^2}{2a^2}\exp(s)\right]^{-1} \tag{6}$$

where $s = aNX_0^2/(2D)$, and $\mathscr{D}$ are the parabolic cylinder functions. We compare the theoretical linear response function with the numerically obtained one in Fig. 3. The qualitative correspondence is good, moreover, the maxima of the curves are rather good reproduced with the formula (6). This shows that the resonant system size is quite good quantitatively described by the Gaussian approximation, see Fig. 3.

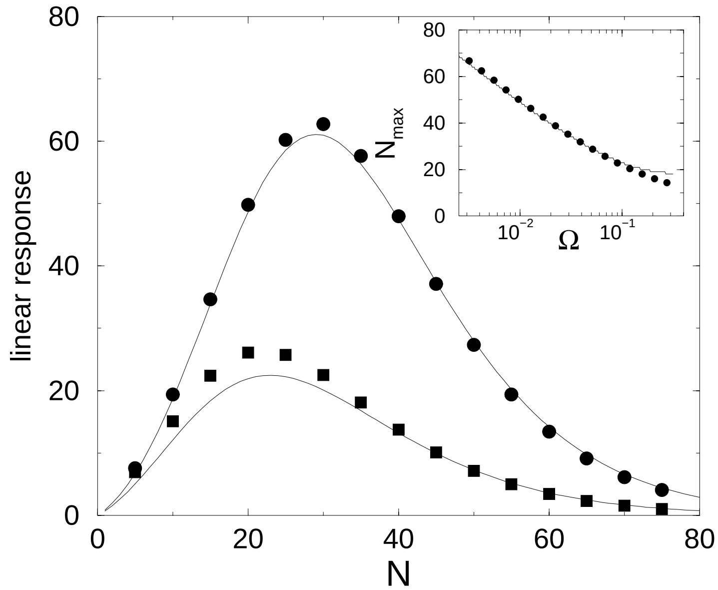

FIGURE 3. Comparison of the system-size dependencies of the linear response function for frequencies $\Omega = 0.05$ (circles) and $\Omega = 0.1$ (squares) with theory (6). The parameters are $D = 1$ and $g - g_c = 2.5$ (where the the exact g_c and the approximate $g_c = 3D$ are used for the ensemble and the Gaussian approximation, respectively). Inlet: Dependence of the system size yielding maximal linear response on the driving frequency Ω (circles: simulations of the ensemble (1), line is obtained by maximizing the expression (6)). Reprinted from [7].

As the last example of the system size resonance we consider a lattice where each individual element does not exhibit bistable noisy dynamics, but such a behavior appears due to interaction and multiplicative noise. This model is described by the set of Langevin equations [20, 21, 22]

$$\dot{x}_i = -x_i(1+x_i^2)^2 + \frac{g}{K}\sum_j (x_j - x_i) + \sqrt{2D}\xi_i(t)(1+x_i^2) + f(t)\,. \tag{7}$$

The difference to the model (1) is that the noise is multiplicative and the on-site potential has only one minimum. K is the number of elements to which the oscillator i is coupled, for global coupling $K = N$ and for a lattice of dimension d with nearest-neighbors coupling $K = 2d$. As has been demonstrated in [20, 21], in some region of couplings g system (7) exhibits the Ising-type transition. If an additional additive noise is added to (7), then one observes transitions between these states and the so-called double stochastic resonance in the presence of the periodic forcing [23, 24]. As is evident from the considerations above, such transitions occur even in the absence of the additive noise if the system is finite. Thus, the system size resonance should be observed in the lattice (7) as well. This is confirmed by numerical calculations presented in [7]. Note that in [24] a realistic electronic circuit modeling the ensemble similar to (7) is described, providing a possible experimental realization of the effect.

SYSTEM SIZE COHERENCE RESONANCE

Here we consider a one-dimensional lattice of N diffusively coupled FitzHugh–Nagumo systems

$$\varepsilon \frac{dx_k}{dt} = x_k - \frac{x_k^3}{3} - y_k + g(x_{k-1} + x_{k+1} - 2x_k) \,, \tag{8}$$

$$\frac{dy_k}{dt} = x_k + a + D\xi_k(t) \,, \qquad 1 \le k \le N \,. \tag{9}$$

We assume periodic boundary conditions and identity of the parameters of the oscillators, only the noise terms ξ_k are different ($\langle \xi_k \rangle = 0$, $\langle \xi_k(t)\xi_m(t') \rangle = \delta_{km}\delta(t-t')$). In [10] it has been numerically and analytically demonstrated, that pulses, generated by noise in a single FitzHugh–Nagumo oscillator, have maximal coherence at a certain noise level. In particular, given the interpulse intervals τ_i the ratio

$$R = \frac{\sqrt{\langle \tau^2 \rangle - \langle \tau \rangle^2}}{\langle \tau \rangle}$$

has been calculated, this quantity measures the closeness of the pulse train to a periodic one (where R obviously vanishes), and as a function of noise intensity D the factor R has a minimum.

Here we investigate the dependence of the factor R on the system size N, keeping the noise and other parameters constant. The results are presented in Fig. 4. The coherence factor R is minimal at $N \approx 10$. We emphasize that we have considered the system (8,9) under very strong coupling and for moderate system length N. This ensured that for considered system sizes (up to 100) all oscillators fired nearly synchronously. Thus the factor R does not depend on which oscillator is chosen for analysis. We expect that the dynamics in the lattice will become much more complex if either the coupling is reduced or the system size is increased, or both (cf. numerical experiments with weakly coupled lattice of FitzHugh–Nagumo oscillators [25]).

CONCLUSIONS

We have demonstrated that in coupled noise-driven oscillators a maximal order can occur at a certain system size. The effect is explained by reduction to the usual phenomena of stochastic and coherence resonances with an effective noise depending on the number of coupled elements. Similar effects have been recently reported for ion channels [26, 27]. One possible field of application of the system size resonance is the

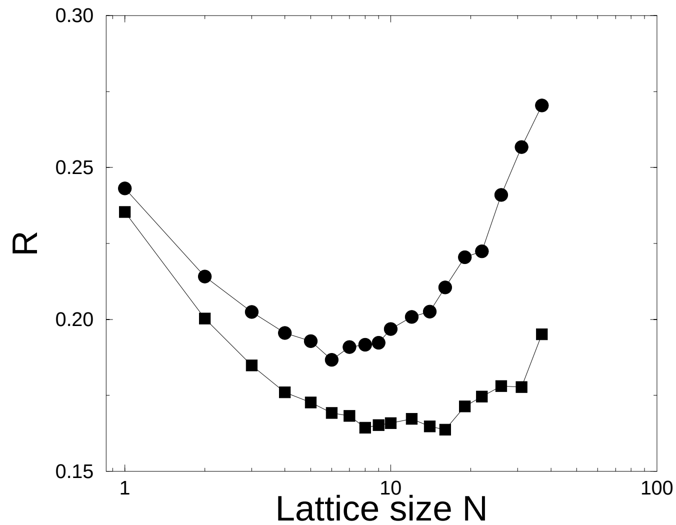

FIGURE 4. Coherence factor R vs. lattice size for the FitzHugh–Nagumo model (8,9) for $\varepsilon = 0.01$, $D = 0.2$, $g = 30$. Circles: $a = 1.05$, squares: $a = 1.03$.

neuronal dynamics. Individual neurons have been demonstrated to exhibit stochastic resonance [28]. While in experiments one can easily adjust noise to achieve the maximal sensitivity to an external signal, it may be not obvious how this adjustment takes place in nature. The above consideration shows, that changing the number of elements in a small ensemble of coupled bistable elements to the optimum can significantly improve the sensitivity. Moreover, changing its connectivity and/or coupling strength, a neuronal system can tune itself to signals with different frequencies.

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