Fluctuation and Noise Letters
Vol. 2, No. 3 (2002) L157-L168

DOUBLY STOCHASTIC EFFECTS

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Received (25 June 2002)
Accepted (20 September 2002)

Doubly stochastic effects are effects in which an optimization of both multiplicative and additive noise intensities is necessary to induce ordering in a nonlinear system. I review recent achievements in the investigation of these effects and discuss two phenomena: doubly stochastic resonance and noise-induced propagation in monostable medium. Finally I discuss possible experimental implementations of these phenomena.

Keywords: Stochastic resonance; multiplicative; additive; noise; doubly stochastic.

1. Introduction

It is not surprising nowadays that noise or random fluctuations can induce counter-intuitive effects, in which noise exhibits a constructive, leading to ordering, role in the behaviour of dynamical systems. Many phenomena have confirmed this ability of noise, among these noise-induced phenomena, one can distinguish between several basic ones, such as noise-induced transitions [1–6], stochastic resonance [7, 8], coherence resonance [9], or stochastic transport in ratchets [10]. The most popular example of noise-induced ordering, which can be also found in the behaviour of biological objects [11] as well as in human recognition [12] or in human brain waves [13], is the effect of stochastic resonance (SR). In the most standard situation SR consists in an optimization by noise of the response of a bistable system to a weak periodic signal. In addition to this conventional situation, due to its generality and universality SR has been found in a large variety of systems, as monostable [14], excitable [15], non-dynamical [16], and thresholdless [17] systems, in systems without an external force (called coherence resonance) [9, 18], and in systems with transient noise-induced structure [19].

Noteworthy, the principle of SR can be also extended for the case of spatially distributed systems. In such systems an optimal intensity of noise may lead to noise-enhanced propagation, in which the propagation of a harmonic forcing through an unforced bistable or excitable medium is increased for an optimal intensity of additive noise [20,21]. This phenomenon has all ingredients characteristic of SR, because the system exhibits locally the noise-induced amplification of a weak periodic signal coming from the neighboring sites. It is important to note that, although numerous works about noise-induced propagation exist (e.g. [22–24]), to our knowledge propagation in monostable media, which is a very important class of dynamical systems, has not been considered before. An interesting exception to this point is the thermal resonance in a signal transmission [25], where noise-induced propagation has been found in monostable systems, but without a local potential and with nonlinear coupling.

In this review we discuss several phenomena in the frame of a concept of doubly stochastic effects, which also demonstrate an improvement of signal processing or signal propagation in nonlinear dynamical systems. This concept has been recently introduced as new mechanism leading to noise-induced ordering in nonequlibrium systems. The idea of this concept is the following. If we observe noise-induced order in a nonlinear system, it occurs due to the presence of some intrinsic property of a system, which together with noise results in noise-induced ordering. For example, in the conventional scenario of SR this feature is a threshold, which is present in the system. Coming noise interacts with this feature, and improves a response of the system to the external periodic signal. Meanwhile, a crucial property of a system, a potential threshold in this case, can be also induced by noise. Usually it happens if we have an interplay of multiplicative and additive noise in the system. In this case multiplicative noise induces a property of a system and additive noise maximizes an ordering in a system with this property. Hence, such effects can be called *doubly stochastic effects* (DSE), because for maximal ordering an optimization of both noise intensities is necessary. Certainly, in such effects an energy of noise is used more efficiently, because it is used not only for noise-induced ordering, but also for system property, which is necessary for this ordering.

In this paper we review two DSE, doubly stochastic resonance (DSR) [26] and noise-induced propagation (NIP) in monostable media [27]. After an introduction of a model and reviewing of noise -induced phase transitions, demonstrated by this model, we describe the effect of DSR. In DSR multiplicative noise (in combination with spatial coupling) induces bistability in a deterministically monostable system, and additive noise induces synchronization with the external signal in this noise-induced bistable regime. Following this, we show that this system can exhibit doubly stochastic effects which lead to signal propagation, if the system is periodically excited from one side. Finally we discuss a possible experimental implementation of suggested theoretical findings in designed simple electronic circuit [28]. In the conclusion we discuss the obtained results and possible directions of the future research.

2. A Model and Noise-induced Phase Transition

We study a general class of spatially distributed systems of elements, which are locally coupled and periodically forced:

$$\dot{x}_i = f(x_i) + g(x_i)\,\xi_i(t) + \frac{D}{4}\sum_{j\in nn(i)}(x_j - x_i) + \zeta_i(t) + A_i\,\cos(\omega t + \varphi)\,, \qquad (1)$$

where x_i is defined in a two-dimensional discrete space of $N\times N$ cells, with i denoting the cell position ($i = i_x + N(i_y - 1)$, where i_x and i_y run from 1 to N). The sum in the right-hand side runs over all nearest neighbors of site i $[nn(i)]$. The additive and multiplicative noise terms are mutually uncorrelated Gaussian distributed with zero mean, and white both in space and time, i.e. $\langle\zeta_i(t)\zeta_j(t')\rangle = \sigma_a^2\delta_{ij}\delta(t-t')$ and $\langle\xi_i(t)\xi_j(t')\rangle = \sigma_m^2\delta_{ij}\delta(t-t')$.

In the absence of periodic forcing ($A_i = 0$), different types of noise-induced phase transitions can be obtained for different forces $f(x_i)$ and $g(x_i)$. In particular, a system with a monostable local deterministic potential can exhibit a phase transition to a noise-induced bistable state [3, 29]. This transition breaks a symmetry and ergodicity of a system and leads to the formation of a non-zero mean field (see Fig. 1). The reason of this phase transition is the common effect of short time bistability induced by multiplicative noise and coupling. To understand which forces $f(x_i)$ and $g(x_i)$ are necessary for the demonstration of noise-induced transition, let us consider the following argumentation [30]. The time evolution of the first moment of a single element can be found by the drift part in the corresponding Fokker-Planck equation (Stratonovich case)

$$\langle\dot{x}\rangle = \langle f(x)\rangle + \frac{\sigma_m^2}{2}\langle g(x)g'(x)\rangle. \tag{2}$$

Next if we start with an initial Dirac δ function, follow it only for a short time, such that fluctuations are small and the probability density is well approximated by a Gaussian. A suppression of fluctuations, performed by coupling, which is absolutely necessary for the transition under consideration, makes this approximation appropriate in our case [31]. The equation for the maximum of the probability, which is also the average value in this approximation $\bar{x} = \langle x\rangle$, has the following form

$$\dot{\bar{x}} = f(\bar{x}) + \frac{\sigma_m^2}{2} g(\bar{x})g'(\bar{x}), \tag{3}$$

which is valid if $f(\langle x\rangle) >> \langle\delta x^2\rangle f''(\langle x\rangle)$. For this dynamics an "effective" potential $U_{\mathrm{eff}}(x)$ can be derived, which has the form

$$U_{\mathrm{eff}}(x) = U_0(x) + U_{\mathrm{noise}} = -\int f(x)dx - \frac{\sigma_m^2 g^2(x)}{4}, \tag{4}$$

where $U_0(x)$ is a monostable potential and U_{noise} represents the influence of the multiplicative noise. If this effective potential is bistable for some intensity of multiplicative noise, then with some approximation the system can undergo noise-induced phase transition, which leads to bistability of the mean field.

More precisely, a borderline of the phase transition can be found analytically by means of the standard mean-field theory procedure [3]. This mean-field approximation is based on replacing the nearest-neighbor interaction by a global term in the Fokker-Planck equation corresponding to (1) for $A_i = 0$. A steady-state solution of Fokker-Planck eq. then gives:

$$w_{\mathrm{st}}(x,m) = \frac{C(m)}{\sqrt{\sigma_m^2 g^2(x) + \sigma_a^2}} \exp\left(2\int_0^x \frac{f(y) - D(y-m)}{\sigma_m^2 g^2(y) + \sigma_a^2}\,dy\right), \tag{5}$$

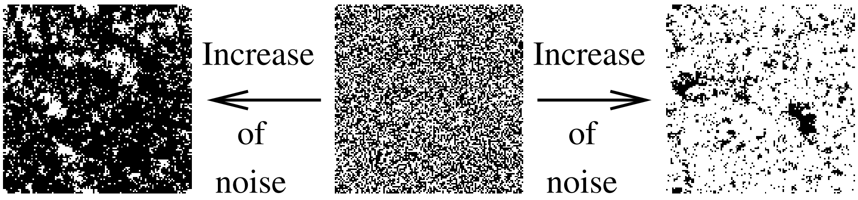

Fig. 1. A visualization demonstration of the phase transition in the model (1). In the disordered phase the mean field is zero due to the random deviation of different elements around zero (middle). In the ordered phase, induced by noise, the symmetry is broken and the mean field is either positive (right) or negative (left). The elements in the lattice 128×128 are coded in accordance to its sign: if positive or zero - white, if negative - black.

where $C(m)$ is a normalization constant and m is the mean field, defined by the equation:

$$m = \int_{-\infty}^{\infty} x w_{\mathrm{st}}(x, m) dx. \tag{6}$$

The self-consistent solution of eq.(6) determines transitions between ordered ($m \neq 0$) and disordered ($m = 0$) phases. Below we consider two examples of functions $f(x)$ and $g(x)$, which provide a possibility of a noise-induced phase transition.

3. Doubly Stochastic Resonance

DSR is a synthesis of noise-induced phase transition and conventional SR. To demonstrate DSR the functions $f(x)$ and $g(x)$ in eq.(1) are taken to be of the form [29]:

$$f(x) = -x(1 + x^2)^2, \qquad g(x) = 1 + x^2. \tag{7}$$

With these forces, a system (1) undergoes a phase transition, whose transition boundaries between different phases are shown in Fig. 2 (left) and the corresponding dependence of the order parameter on σ_m^2 is presented in Fig. 2 (right).

Next we consider how the system (1) responses to the global periodic forcing ($A_i = A$). We have taken a set of parameters ($\sigma_m^2; D$) within the region of two coexisting ordered states with a nonzero mean field. In particular, we choose values given by the dot in Fig. 2 (left). As for the network, we take a two-dimensional lattice of $L^2 = 18 \times 18$ oscillators, which is simulated numerically [32] with a time step $\Delta t = 2.5 \times 10^{-4}$ under the action of the harmonic external force. The amplitude of the force A has to be set sufficiently small to avoid hops in the absence of additive noise. Jumps between $m_1 \leftrightarrow m_2$ occur only if additive noise is additionally switched on. Runs are averaged over different initial phases. Time series of the mean field and the corresponding periodic input signal are plotted in Fig. 3 left for three different values of the intensity of additive noise σ_a^2. The current mean field is calculated as $m(t) = \frac{1}{L^2} \sum_{i=1}^{N} x_i(t)$. For small σ_a^2, hops between the two symmetric states m_1 and m_2 are rather seldom and not synchronized to the external force. If we increase the intensity σ_a^2, we achieve a situation when hops occur with the same periodicity as the external force and, hence, the mean field follows with high probability the input force. An increase of additive noise provides a synchronization of the output of the

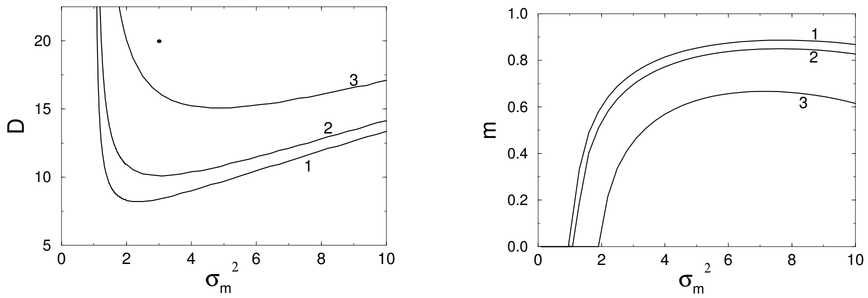

Fig. 2. Left: Boundaries of the bistable regime on the plane $(\sigma_m^2; D)$ for different intensities of the additive noise $\sigma_a^2 = 0$ (1); 1 (2), and 5 (3). The black point corresponds to $D = 20$, $\sigma_m^2 = 3$. Right: The order parameter $|m|$ vs the intensity of the multiplicative noise for $D = 20$ and $\sigma_a^2 = 0$ (label 1), 1 (label 2), and 5 (label 3). Inside the ordered region for fixed value of σ_m^2 an increase of the additive noise intensity leads to the decrease of the order parameter.

system with input forcing. If σ_a^2 is increased further, the order is again destroyed, and hops occur much more frequently than the period of the external force. Note also that for large σ_a^2 the value of the mean field which corresponds to the stable state is becoming smaller. It is caused by the fact that additive noise influences also transition lines. An increase of σ_a^2 results in a reduction of the ordered region (Fig. 2 (left), curves 2 and 3) and decreasing the value $m_1 = -m_2$ (Fig. 2 (right), curves 2 and 3).

To quantify this DSR-effect, we have calculated the signal-to-noise ratio (SNR) by extracting the relevant phase-averaged power spectral density $S(\omega)$ and taking the ratio between its signal part with respect to the noise background [8]. The dependence of SNR on the intensity of the additive noise is shown in the Fig. 3 (right) for the mean field (filled points) and the mean field in a 2-state approximation (opaque point). In this 2-states approximation we have replaced $m(t)$ by its sign and put $m(t) = +1$ or $m(t) = -1$, respectively. Both curves exhibit the well known bell shaped dependence on σ_a^2 typically for SR. Since the bimodality of the mean field is a noise-induced effect we call that whole effect *Doubly Stochastic Resonance.* For the given parameters and $A = 0.1$, $\omega = 0.1$ the maximum of the SNRs is approximately located near $\sigma_a^2 \sim 1.8$.

To obtain analytic estimates of the SNR, an approximation of "effective" potential (4) can be used. For this we consider a conventional SR problem in this potential with an external periodic force of the amplitude A and the frequency ω. If we neglect intrawell dynamics and follow linear response theory the SNR is well known [8, 33]

$$SNR_1 = \frac{4\pi A^2}{\sigma_a^4} r_k \tag{8}$$

where r_k is the corresponding Kramers rate [34]

$$r_k = \frac{\sqrt{(U''_{\text{eff}}(x)|_{\text{x=x}_{\text{min}}} |U''_{\text{eff}}(x)|_{\text{x=x}_{\text{max}}})}}{2\pi} \exp(-\frac{2\Delta U_{\text{eff}}}{\sigma_a^2}) \tag{9}$$

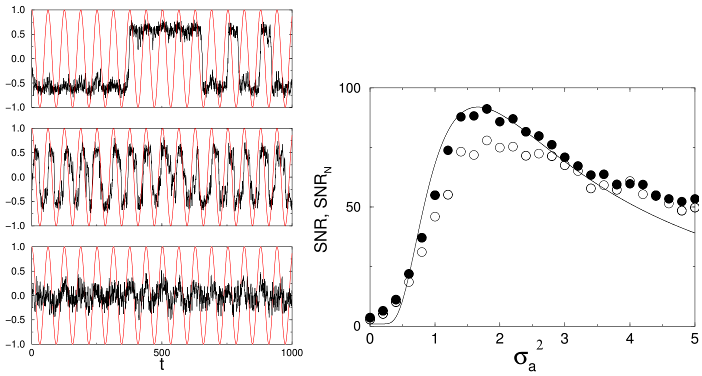

Fig. 3. Left: The time evolution of the current mean field (output) and the periodic external force $F(t)$ (input) for different intensities of additive noise (from top to bottom) $\sigma_a^2 = 0.01$, 1.05, and 5.0. If the intensity of the additive noise is close to their optimal value (middle row), an input/output synchronization occurs. The remaining parameters are: $A = 0.1$, $\omega = 0.1$, $D = 20$, and $\sigma_m^2 = 3$. Right: The dependence of SNR on the additive noise intensity for the output (filled points) and its 2-states approximation (opaque points). The solid line corresponds to the analytical estimation $SNR_{\rm N}$ (10). The processing gain is $G = 0.7$.

for surmounting the potential barrier $\Delta U_{\rm eff}$. Using Eqs.(4),(8), and (9) we get an analytical estimates for a single element inside the lattice. Further on, rescaling this value by the number N of oscillators in the lattice [35] and taking into account the processing gain G and the bandwidth Δ in the power spectral density [33, 36, 37], the $SNR_{\rm N}$ of the mean field of the network of N elements can be obtained

$$SNR_{\rm N} = SNR_1 \frac{NG}{\Delta} + 1. \qquad (10)$$

This dependence is shown in the Fig. 3 right by the solid line and demonstrates despite the rough approximation a good agreement with the results of the numerical simulations. Nearly exact agreement is found in the location of the maximum as well as for the quantitative values of the SNR ("scalloping loss" [33] has been avoided in simulations by setting the frequency ω to be centered on one of the bins in the spectrum).

4. Noise-induced Propagation in Monostable Media

Next we study a propagation in the system (1). In this case the periodic forcing is applied to the system (1) coherently along only one side, as shown in Fig. 4 (left) $[A_i = A(\delta_{i_x,1} + \delta_{i_x,2} + \delta_{i_x,3})]$. Even though the results shown below are very general, for a quantitative study we choose particular functions $f(x) = -f_1(x)$(see eq.(12)) and $g(x) = x$ [28]. Regions of bistability can be as above determined approximately by means of a standard mean-field procedure [3] and are shown in Fig. 4(right) in the $D - \sigma_m^2$ plane for three different values of the additive noise intensity.

Now we place ourselves within the bistable regime supported by multiplicative noise and coupling (e.g. $D = 3$, $\sigma_m^2 = 3$), and investigate the propagation of a wave through the system. The boundary conditions are periodic in the vertical direction

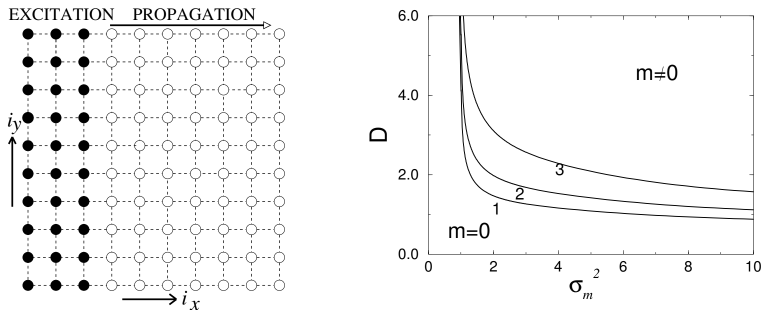

Fig. 4. Left: A lattice which is excited only from one side: elements under the direct periodic action are denoted by black; the first three columns ($i_x = 1, 2, 3$) are periodically driven; all oscillators are under the influence of noise. Right: Mean-field transition lines between disordered monostable ($m = 0$) and ordered bistable ($m \neq 0$) phases for model (see Sec. 3): $\sigma_a^2 = 0.3$ (label 1), $\sigma_a^2 = 0.5$ (label 2) and $\sigma_a^2 = 1.0$ (label 3). Other parameters are $G_a = 0.5$, $G_b = 10$ and $B_p = 1$.

and no-flux in the horizontal direction. The propagation will be quantified by the system's response at the excitation frequency, computed as $Q^{(j)} = \sqrt{{Q_{\sin}^{(j)}}^2 + {Q_{\cos}^{(j)}}^2}$, with

$$Q_{\sin}^{(j)} = \frac{\omega}{n\pi} \int_0^{2\pi n/\omega} m_j(t) \sin(\omega t) dt \, , Q_{\cos}^{(j)} = \frac{\omega}{n\pi} \int_0^{2\pi n/\omega} m_j(t) \cos(\omega t) dt \, , \qquad (11)$$

where $m_i(t)$ is the field (voltage) averaged along the vertical column (Fig. 4), i.e. $m_j(t) = \frac{1}{N} \sum_{k=1}^{N} x_{i+(k-1)N}(t)$.

The value of $Q^{(j)}$ for different oscillators along the chain is shown in Fig. 5(a) for increasing intensities of additive noise within the noise-induced bistable regime. The forcing amplitude is taken to be large enough to produce hops between the two wells in the bistable oscillators, without a need of additive noise. Therefore, for the first oscillators an increase of additive noise leads only to a decreasing response at the forcing frequency, whereas for distant oscillators the situation changes qualitatively. There, a response is induced that depends non-monotonically on the additive noise intensity. Clearly, a certain amount of additive noise exists for which the propagation of the harmonic signal is optimal. For smaller multiplicative-noise intensity [Fig. 5(b)] the system leaves the bistable region; hence the response is small and always monotonically decreasing. Hence, the resonant-like effect requires suitable intensities of *both* the additive and multiplicative noises.

A propagation of the harmonic signal can also be obtained for values of the forcing amplitude small enough so that hops are not produced in the directly excited sites in the absence of additive noise. This is the regime in which DSR really occurs in the excited part of the system, and the excitation propagates through the rest of the lattice enhanced by noise. Now all the oscillators have a non-monotonic dependence on the additive noise intensity for a multiplicative noise within the bistable region [Fig. 5(c)], and a monotonic one for a multiplicative noise within the monostable region [Fig. 5(d)]. The former case corresponds to a spatiotemporal propagation in the DSR medium, and we call this phenomenon *spatiotemporal doubly stochastic resonance* (SDSR).

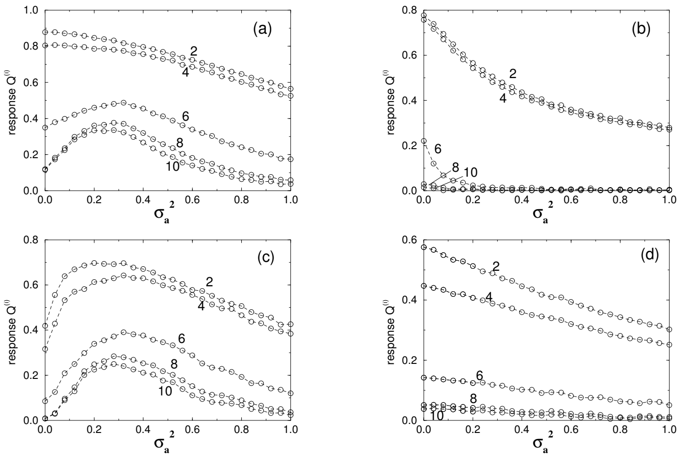

Fig. 5. Response $Q^{(j)}$ to a periodic excitation in different oscillators (the order j is shown in the curve labels) vs. additive-noise intensity (a,c) inside the bistability region ($\sigma_m^2 = 3$), and (b,d) outside that region ($\sigma_m^2 = 0.5$). As shown in Fig. 1, the oscillators with index $j = i_x = 1, 2, 3$ are directly excited by the periodic force, and oscillators with $j = i_x > 3$ are excited through the excitation propagation. Parameters are those of Fig. **??**, and $D = 3$. The amplitude is: (a,b) $A = 0.3$ (noise-induced propagation) and (c,d) $A = 0.2$ (spatiotemporal doubly stochastic resonance).

Using an approximation of "effective"potential this effect can be understood in the frame of a standard SR mechanism [8], where the external signal is provided by the periodic force for the directly excited oscillators, and by the influence of the left neighbors for the non-excited oscillators. For large forcing, only the latter need an additive noise to hop synchronously between the wells, whereas for small forcing, both the excited and the non-excited oscillators display SR. These two behaviors correspond to the situations depicted in Figs. 5(a) and 5(c), respectively.

5. Experimental Implementation

We expect that these theoretical findings will stimulate experimental works to verify DSR in real physical systems (for the first experimental observation of noise-induced bistability see [38]). Appropriate situations can be found in electronic circuits [**?**], as well as in systems, which demonstrate a noise-induced shift of the phase transition, e.g, in: liquid crystals [39, 40], electronic cellular neural networks [41–43], photosensitive chemical reactions [44, 45], or Rayleigh-Bénard convection [46]. It can be crucial for such experiments, that in doubly stochastic effects the energy of noise is used in a more profitable way: not only for the optimization of the signal processing or propagation, but also for the support of the potential barrier to provide this optimization.

Here we discuss a design of a simple electronic circuit which can be used for the demonstration of these phenomena [28]. This electrical circuit consists of N coupled elements (i, j). A circuit of one element is shown in Fig. 6 (a). Three ingredients in this circuit are important: input current, time-varying resistor (TVR) and a nonlinear resistor. Every element is coupled with its neighbours by the resistor R_c (i.e. by diffusive coupling). The capacitor is shown by C. The nonlinear resistor R_N can be realized with a set of ordinary diodes [47, 48], whose characteristic function is a piecewise-linear function

$$i_N = f(V) = \begin{cases} G_b V + (G_a - G_b)B_p & \text{if } V \leq -B_p, \\ G_a V & \text{if } |V| < B_p, \\ G_b V - (G_a - G_b)B_p & \text{if } V \geq B_p, \end{cases} \tag{12}$$

where i_N is the current through the nonlinear resistor(R_N), V is the voltage across the capacitor(C), and parameters G_a, G_b and B_p determine the slopes and the breakpoint of the piecewise-linear characteristic curve. The next important ingredient is a time-varying resistor(TVR) [48, 49]. The conductance $G(t)$ of TVRs varies with time. Presently, we consider the case that the function which represents the variation of the TVR is a Gaussian $\delta-$correlated in space and time noise, i.e. $G(t) = \xi(t)$, where

$$\langle \xi_i(t)\xi_j(t')\rangle = \sigma_m^2 \delta_{i,j}\delta(t-t').$$

An external action on the elements under direct excitation in the circuit is performed by the current input $I(t)$, which is a periodic signal (with amplitude A, frequency ω, and initial phase φ), additively influenced by independent Gaussian noise $\zeta(t)$ with intensity σ_a^2

$$I(t) = \zeta(t) + A_i\cos(\omega t + \varphi), \quad \langle \zeta_i(t)\zeta_j(t')\rangle = \sigma_a^2 \delta_{i,j}\delta(t-t').$$

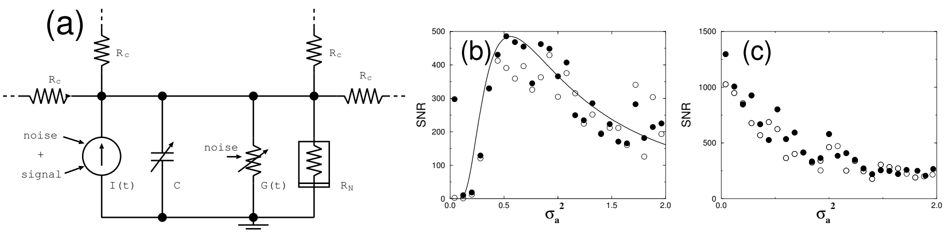

Fig. 6. (a) The electronic circuit of the element (i, j). (b) Numerical SNR (points) vs analytical estimation (solid line) for the equation with f_1 and $D = 3$, $\sigma_m^2 = 3$. Numerical results are shown by black points for the mean field and opaque points for its two-state approximation. The stochastic resonance effect is supported by noise. If we decrease the intensity of multiplicative noise, we do not observe it; e.g. for (c) $D = 3$, $\sigma_m^2 = 0.5$.

The electronic circuit with respect to the element (i, j) can be described by a set of Kirchoff's equations:

$$C\frac{dV_{i,j}}{dt} = I(t) - G(t)V_{i,j} - f(V_{i,j}) + \frac{1}{R_c}(V_{i+1,j} + V_{i-1,j} + V_{i,j+1} + V_{i,j-1} - 4V_{i,j})$$

Hence, the following set of Langevin equations describes the considered system:

$$\begin{aligned} \frac{dV_{i,j}}{dt} &= -f(V_{i,j}) + V_{i,j} + \zeta_{i,j}(t) + A_i\cos(\omega t + \varphi)\xi_{i,j}(t) \\ &+ \frac{D}{4}(V_{i+1,j} + V_{i-1,j} + V_{i,j+1} + V_{i,j-1} - 4V_{i,j}), \end{aligned} \tag{13}$$

where C is set to unity by normalization of time and D denotes a strength of coupling equal to $\frac{4}{CR_c}$. In the case when f_2 represents the TVR, the model is the time-dependent Ginzburg-Landau equation, which is a standard model to describe phase transitions and critical phenomena in both equilibrium and nonequilibrium situations [3]. It is important that we consider only the situation when the potential of one element is monostable ($G_a = 0.5$, $G_b = 10$, $B_p = 1$), hence avoiding the possibility to observe SR without multiplicative noise.

Due to the noise-induced bistability (transitions boundaries are shown in the Fig. 4 right), this circuit will demonstrate both DSE effects considered above. We focus on the case of DSR, i.e. an external excitation is applied to each element and $A_i = A$. An analytical estimation of DSR effect, calculated as in Sec. 2, is shown in the Fig. 6 (b). The DSR effect is clearly observed in the behaviour of SNR of the output mean voltage vs. the intensity of additive noise. To verify the analytical results numerically, we have also performed simulations of the model (13. We have taken a set of parameters within the region of two coexisting ordered states with nonzero mean field. As a total system, we take a two dimensional lattice of 18×18 elements, which was simulated numerically with a time step $\Delta t = 2.5 \times 10^{-4}$. The amplitude of the external signal was set to 0.1, i.e. sufficiently small to avoid hops between two states in the absence of additive noise. The numerically obtained dependence of SNR on the intensity of the additive noise is shown in Fig. 6(b) for the mean-field (filled points) and the mean field in a two-state approximation (opaque points). In this two-state approximation, we have replaced the value of the mean field in time-series by its sign before calculating the power spectral density. Both curves demonstrate well-known bell-shaped dependence which is typical for SR. Let us note, that for these version of the model SNR for the mean field tends to infinity for small values of additive noise intensity (see black points for $\sigma_a^2 < 0.1$). Numerical simulations agree very good with our theoretical estimation despite the very rough approximation via "effective" potential (we will study the question, what is the parameters regions of its validity, in a following paper).

The fact that this SR effect is created by multiplicative noise, can be illustrated as the following. If we decrease only the intensity of multiplicative noise, other parameters fixed, the SR effect is not observed, as it is shown in Fig. 6(c). The reason is that in this case our system is not bistable (see Fig. 4 right) For experimental setup a minimal number of elements, which are necessary for DSR observation, can be important. Reduction of the elements number in this system leads to the fact, that a system can spontaneously (even in the absence of forcing) perform a hop between two states. These jumps hide DSR effect, since they destroy a coherence between input and output. For the system size 18×18, considered here, such jumps are rather seldom and do not hinder DSR. Our calculations have shown that a size 10×10 is still satisfactory, whereas further decrease of the elements number will destroy the effect.

6. Summary and Outlook

I have reviewed recent findings on doubly stochastic effects. I have considered two doubly stochastic phenomena, DSR and noise-induced propagation in monostable media. In these phenomena the role of noise is twofold: first multiplicative noise (together with coupling) induces a bistability in the spatially distributed system, and then additive noise optimizes a processing or propagation of the input signal. An optimization of both noise intensities is necessary for the demonstration of these doubly stochastic phenomena. Noteworthy, DSR and NIP in monostable media, considered here, differ substantially from the conventional SR and different variations of spatiotemporal SR or NIP in bistable or excitable systems (see discussion in [26, 27]).

One can distinguish between two possible directions of future research on doubly stochastic effects. First, one can search for doubly stochastic effects in other classes of systems, or doubly stochastic effects, which lead to noise-induced ordering of other type. For example, we are going to study doubly stochastic coherence in excitable systems [50], where ordering means a generation of a coherent output in neuron systems. Second, we hope that our theoretical findings will encourage observers to perform experiments to study doubly stochastic effects. Here we have suggested a simple electronic circuit as a possible experimental implementation of doubly stochastic effects, in [26, 27] we have discussed other appropriate experimental situations. We hope that due to its generality the concept of doubly stochastic effects will be confirmed by experiments and used in applications, especially in signal processing systems, such as communication systems or neuron populations.

Acknowledgments

I thank Prof. J. Kurths for extensive discussions and different help. This study was made possible in part by grants from the Microgravity Application Program/Biotechnology from the Manned Spaceflight Program of the European Space Agency (ESA).

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