Additive noise in noise-induced nonequilibrium transitions

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We study different nonlinear systems which possess noise-induced nonequlibrium transitions and shed light on the role of additive noise in these effects. We find that the influence of additive noise can be very nontrivial: it can induce first- and second-order phase transitions, can change properties of on–off intermittency, or stabilize oscillations. For the Swift–Hohenberg coupling, that is a paradigm in the study of pattern formation, we show that additive noise can cause the formation of ordered spatial patterns in distributed systems. We show also the effect of doubly stochastic resonance, which differs from stochastic resonance, because the influence of noise is twofold: multiplicative noise and coupling induce a bistability of a system, and additive noise changes a response of this noise-induced structure to the periodic driving. Despite the close similarity, we point out several important distinctions between conventional stochastic resonance and doubly stochastic resonance. Finally, we discuss open questions and possible experimental implementations. [DOI: 10.1063/1.1380369]

In the majority of investigations, devoted to the study of noise-induced processes, a supplement of additive noise leads only to smoothing of transition diagrams. Contrary to this situation, in this contribution we show that additive noise can play a much more crucial role. In oscillatory systems, additive noise is able to excite oscillations, to influence on–off intermittency, and to stabilize stochastic oscillations. In spatially extended systems, which consist of coupled overdamped oscillators, additive noise can induce first- and second-order phase transitions, which in particular cases manifest themselves in the appearance of spatially ordered patterns. Another interesting behavior occurs if a system works as a signal processor. Then additive noise is able to optimize the response of a system to an external periodic signal, if this system possesses a property of multiplicative noise induced bistability.

I. INTRODUCTION

Intensive investigations in nonlinear physics in the last two decades have shown that there are many nonequilibrium systems which demonstrate phenomena manifesting noise-induced ordering. Among these phenomena we emphasize several basic ones, such as stochastic resonance (SR)[1,2] (for SR in natural systems see Ref. 3), noise-induced transitions (NIT),[4–6] noise-induced transport in ratchets,[7] or coherence resonance.[8] This classification does not pretend to be complete, because there are various modifications and extensions of these basic phenomena (e.g., resonance activation[9] or noise-induced pattern formation[10]). On the another hand, there are phenomena which possess properties of different groups from this classification. Two interesting examples may illustrate this point: a synthesis of a ratchet mechanism and noise-induced phase transition,[11] and a synthesis of stochastic resonance and noise-induced transition.[12]

In the present review we focus on one of these basic phenomena, namely noise-induced transitions (NIT). In its turn, NIT can be classified into three main groups: (i) NIT which lead to the appearance of additional maxima in the system's probability distribution,[4] (ii) NIT which lead to the excitation of oscillations,[13,14] and (iii) NIT in extended systems which lead to breaking of symmetry and the creation of a mean field.[5,15–17,19] In the majority of the papers on these topics only multiplicative noise is perceived to be responsible for the transitions. However, it was recently shown,[6,18,20–22] that under certain conditions additive noise can also be very important and nontrivial in NIT. The aim of the present paper is to discuss several aspects and recent results of this investigation and also to point out open questions and unsolved problems connected with the influence of additive noise on transitions in nonlinear systems.

First we analyze *oscillatory* systems under the action of noise. In Sec. II we start by considering a transition induced by multiplicative noise in a pendulum with randomly vibrated suspension axis. We investigate the role of additive noise in this effect and show that additive noise influences a transition as well as on–off intermittency, observed in the excited oscillations. In contrast to this situation in which additive noise only smoothes the transition, in the next investigated oscillatory model, a standard epidemiological model with random excitation, the transition can be induced both by multiplicative and additive noise (Sec. III). Moreover, additive noise is able to stabilize stochastic oscillations, which are unstable if only multiplicative noise is present. Another class of models under consideration are spatially extended systems, which consist of coupled *overdamped* oscillators. We show that in such systems second- and first-order transitions induced by additive noise are possible (Sec. IV). If a nonlinear distributed system is under the action of additional external force, then doubly stochastic resonance (DSR) can be observed (Sec. V). In DSR the influence of noise is twofold: multiplicative noise induces a bistability of a mean field, and additive noise helps the system to respond coherently to an external signal. Finally, we summarize the results and discuss open questions of the problem under consideration in order to show that there are a lot of unsolved problems in this particular field, which is rapidly developing and attracting constantly growing attention in the modern nonlinear physics.

II. TRANSITIONS IN THE PRESENCE OF ADDITIVE NOISE: ON–OFF INTERMITTENCY

A pendulum with randomly vibrated suspension axis is a typical example of oscillatory system, in which parametric action of noise can lead to the excitation of oscillations via a second-order phase transition.[6,13,18] In this case the intensity of multiplicative noise plays the role of temperature and the average amplitude is the order parameter. Here we discuss the question "what happens if additionally additive noise is acting upon the system?" Therefore we consider a pendulum whose suspension axis is vibrating in the direction making the angle γ with respect to the vertical. As shown in Ref. 6, for moderately small vibrations of a suspension axis, the equation of motion can be written as follows:

$$\ddot{\varphi}+2\beta(1+\alpha\dot{\varphi}^2)\dot{\varphi}+\omega_0^2(1+\xi_1(t))\varphi=\omega_0^2\xi_2(t), \tag{1}$$

where φ is the pendulum angular deviation from the equilibrium position, ω_0^2 is the natural frequency of small free pendulum oscillations, β is a damping factor with the nonlinear coefficient α, $\xi_1(t)=\xi(t)\cos\gamma$ is the multiplicative component of the suspension vibration, and $\xi_2(t)=-\xi(t)\sin\gamma$ is its additive component, $\xi(t)$ is a comparatively wide-band random process (or white noise), responsible for the shift of the suspension axis in the direction of vibration.

In the absence of additive noise ($\xi_2=0$, $\gamma=0$, i.e., a vibration is performed strictly in the vertical direction), the system can be analyzed analytically. Looking for the solution in the form $\varphi(t)=A(t)\cos(\omega_0 t+\phi)$ and using the Krylov–Bogolyubov method for stochastic equations,[23] we obtain the following truncated equations for the amplitude A and the phase ϕ of the pendulum's oscillations:

$$\begin{aligned}\dot{u}&=\frac{1}{8}\omega_0^2\kappa(2\omega_0)-\beta\left(1+\frac{3}{4}\beta\alpha\omega_0^2\exp^{2u}\right)+\frac{\omega_0^2}{2}\zeta_1(t),\\ \dot{\phi}&=\omega_0\zeta_2(t),\end{aligned} \tag{2}$$

where $u=\ln A$, $\zeta_1(t)$, and $\zeta_2(t)$ are white noise with intensities

$$K_1=\tfrac{1}{2}\kappa(2\omega_0),\quad K_2=\tfrac{1}{4}(\kappa(0)+\tfrac{1}{2}\kappa(2\omega_0)). \tag{3}$$

Here $\kappa(\omega)=\int_{-\infty}^{\infty}\langle\xi(t)\xi(t+\tau)\rangle\cos(\omega\tau)d\tau$ is the power spectrum density of the process $\xi(t)$ at the frequency ω, and the angular brackets signify averaging over statistical ensemble. It is important that in the equation for the amplitude $u=\ln A$, we have a constant term $\omega_0^2\kappa(2\omega_0)/8$, which appeared due to the parametric action of noise. Namely, this fact is responsible for the excitation of noise-induced oscillations.

Solving the Fokker–Planck equation associated with Eq. (1), a probability density for the amplitude $w(A)$ and amplitude squared $w(A^2)=(1/2A)\,w(A)$ can be found.[13] Using the function $w(A^2)$ we obtain

$$\langle A^2\rangle=\begin{cases}\dfrac{4\eta}{3\alpha\omega_0^2} & \text{for}\quad \eta\geqslant 0,\\ 0 \quad \text{for}\quad \eta\leqslant 0,\end{cases} \tag{4}$$

where

$$\eta=\frac{\omega_0^2}{8\beta}\left(\kappa(2\omega_0)-\frac{8\beta}{\omega_0^2}\right)$$

is proportional to the difference between the noise intensity at the frequency $2\omega_0$ and the critical noise intensity.

It is clear from this that for $\eta\geqslant 0$ the parametric excitation of the pendulum's oscillations occur under the effect of noise via a noise-induced transition. This manifests itself in the fact that the mean value of the amplitude squared becomes different from zero. The corresponding dependence of the order parameter $\langle A^2\rangle$ on the parameter η is plotted in Fig. 1(a). Numerical simulation of the original Eq. (1) shows that if the noise intensity is slightly over a threshold, then on–off intermittency can be observed in the form of oscillations.[24] This means that for the same external action the system is sometimes in the state "on" (the amplitude is large), which is intermittent with the state "off" (the amplitude is rather small).

Now let us discuss which changes happen in the presence of additive noise. The analytical consideration for this case can be found in Ref. 22; here we present the results of numerical simulations. The results are shown in Fig. 1(a). The presence of additive noise leads to the fact that the probability distribution below the threshold is no longer a δ-function, and the transition is now smoothed and not so well-defined, as in the case without additive noise. It is interesting to note that in both cases, with or without additive noise, no additional extrema in the system probability distribution $w(A^2)$ are observed in the course of the transition.

The additive noise also influences the effect of on-off intermittency [see Fig. 1(b)]. For supercritical values of the multiplicative noise intensity on–off intermittency is now hidden and not observable in the form of oscillations, but can be detected for subcritical values, below a threshold. Hence in the presence of additive noise on–off intermittency, a sign of noise-induced transition, can be observed even before this transition occurs with respect to the increase of the control parameter.

It is necessary to note that in the same system chaotic oscillations can be observed, if the external parametric action is periodic. A comparison with this case is discussed in Ref. 13. Chaotic pendulum's oscillations are very similar in its form to noise-induced oscillations. However, a calculation of the probability distribution of the average amplitude squared allows to distinguish between both cases of the external action by means of the Rytov–Dimentberg criterion.[13]

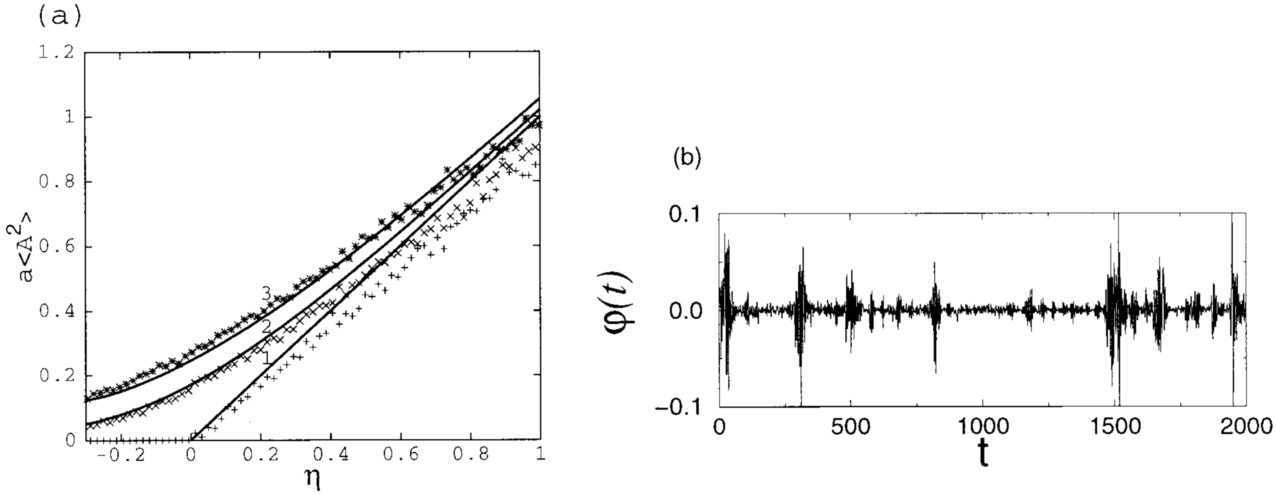

FIG. 1. (a) A noise-induced phase transition in a pendulum with randomly vibrated suspension axis [Eq. (1)]. The dependence of the averaged amplitude squared multiplied by the parameter $a=3\alpha\omega_0^2/4$ on η, where η is an extent on which multiplicative noise intensity exceeds the threshold value. The curve 1 corresponds to the case without additive noise, curves 2 and 3 to the cases with additive noise intensities $k1$ and $k2$, where $k2>k1$ (for details and analytical expressions see Ref. 6). Analytical and numerical results are shown by solid and symbol curves, respectively. (b) On–off intermittency for subcritical values of multiplicative noise intensity. In contrast to this situation, if additive noise is absent, on–off intermittency is observed near a threshold but for supercritical values of the multiplicative noise intensity.

As is shown by further examples in this contribution, this effect of transition smoothing and influence on on–off intermittency is not a single effect of additive noise in oscillatory systems.

III. TRANSITIONS INDUCED BOTH BY MULTIPLICATIVE AND ADDITIVE NOISE: STABILIZATION OF NOISE-INDUCED OSCILLATIONS

In this section we study a system under the action of noise, which has both additive and multiplicative components. We show that these both multiplicative and additive components of noise, considered separately, can induce a transition, and, what is especially interesting, the combination of their actions stabilizes noise-induced oscillations. To demonstrate these effects, we use a standard epidemiological model for the dynamics of children diseases.[25] Two variants of excitation are possible, either by periodic force[26,27] or by noise.[14] In both cases this system exhibits chaotic or noise-induced oscillations which closely resemble oscillations observed in experimental data.

We analyze the influence of additive component of noise in the following model system:[14]

$$\dot{S}=e(1-S)-bSI, \quad \dot{E}=bSI-(e+l)E,$$
$$\dot{I}=lE-(e+g)I, \tag{5}$$

where S, E, and I denote the number of susceptible, exposed but not yet infected, and invective children, respectively. The parameters $1/e$, $1/l$, $1/g$ are the average expectancy, latency and infection periods of time. The contact rate b is the parameter of excitation and equal to $b=b_0(1+b_1\xi(t))$ where $\xi(t)$ is a harmonic noise with the peak of spectral density at the circle frequency 2π (seasonal noisy oscillations with a period equal to one year) and the parameter b_1 is the amplitude of noise. The excited oscillations are executed in the vicinity of the stable singular point with the coordinates (S_0,E_0,I_0). Hence, one can easily rewrite the equations for the new variables $x=S/S_0-1$, $y=E/E_0-1$, and $z=I/I_0-1$ which are deviations from the equilibrium point:

$$\dot{x}+ex=-b_0I_0(1+b_1\xi(t))(x+z+xz)-b_0b_1I_0\xi(t),$$
$$\dot{y}+(e+l)y=(e+l)(1+b_1\xi(t))(x+z+xz)+(e+l)b_1\xi(t), \tag{6}$$
$$\dot{z}+(e+g)z=(e+g)y.$$

This form of equations clearly shows that the action of noise is multiplicative as well as additive.

An increase of the noise intensity causes noise-induced oscillations of the variables S, I, E [Fig. 2(a)]. Their oscillatory behavior closely resembles observed epidemiological data [compare Fig. 2(a) with figures in Ref. 28]. These oscillations are excited after a noise-induced transition [see Fig. 2(b)]. There the variance of oscillations together with an approximating straight line is shown. The point where the straight line crosses the abscissa axis can be taken as a critical point of the transition. To prove this, we remove artificially the multiplicative component of noise from Eqs. (6). In this case the variance of oscillations is equal to zero if $b_1<b_{1cr}$ and goes to infinity shortly after the noise intensity exceeds its critical value. So, additive noise indeed is able to induce a phase transition. The same situation happens if the additive component of noise is absent but the multiplicative one is present.

To conclude, this transition can be induced by noise which contains both multiplicative and additive components. As shown by its separate consideration, both components play an important role in this transition. What is even more interesting, if the additive and multiplicative components of noise act together, as in the model, a stabilization of noise-induced oscillations occurs: in this case the dependence of

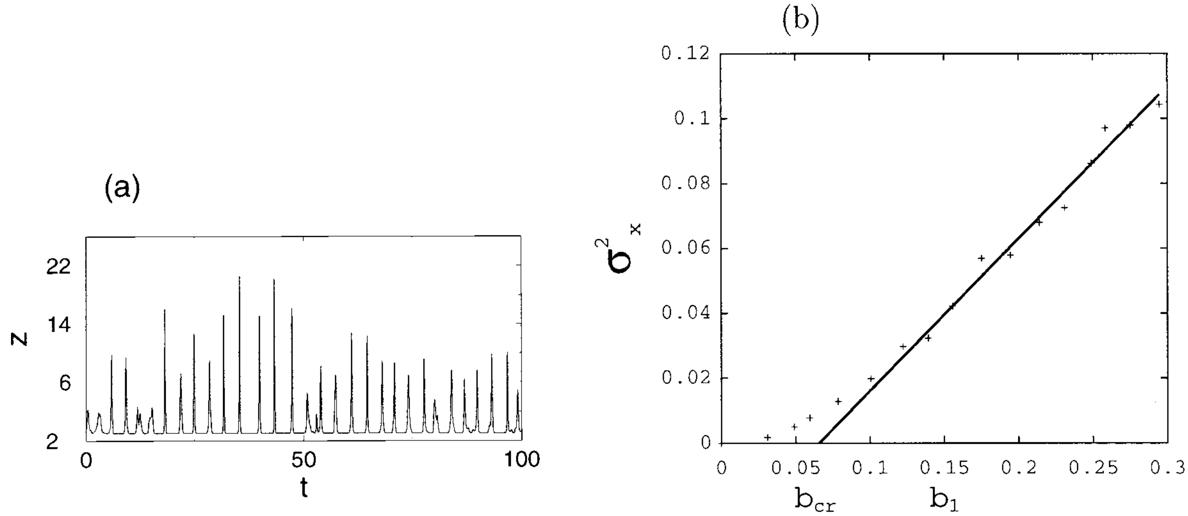

FIG. 2. (a) Noise-induced oscillations (epidemics) in the epidemiological model Eqs. (6). (b) The dependence of oscillation variance for the variable x on the parameter b_1, which is responsible for the noisy variation of a contact rate (see the text).

the variance on the noise intensity does not increase to infinity, that is not a case if multiplicative component of noise acts separately.

IV. TRANSITIONS INDUCED BY ADDITIVE NOISE

Now we extend our study to spatially extended systems and show that additive noise is able to induce second- and first-order phase transitions. Due to a special form of coupling these transition can also lead to the formation of spatially ordered patterns.

A. Second-order phase transitions: Noise-induced pattern formation

We investigate a nonlinear lattice of overdamped coupled stochastic oscillators[10,21] under the action of noise. In this system a transition manifests itself in the formation of spatially ordered patterns, as a consequence of a special form of coupling *a la* Swift–Hohenberg. The system is described by a scalar field $x_{\mathbf{r}}$, defined on a spatial lattice with points $\mathbf{r}$:

$$\dot{x}_{\mathbf{r}} = f(x_{\mathbf{r}}) + g(x_{\mathbf{r}})\xi_{\mathbf{r}} + \mathcal{L}x_{\mathbf{r}} + \zeta_{\mathbf{r}} \tag{7}$$

with f and g taken in the form (for the discussion, which functions can be chosen to observe a transition see Ref. 29)

$$f(x) = -x(1+x^2)^2, \quad g(x) = a^2 + x^2 \tag{8}$$

and $\xi_{\mathbf{r}}$, $\zeta_{\mathbf{r}}$ are independent zero-mean-value Gaussian white noises:

$$\begin{aligned} \langle \xi_{\mathbf{r}}(t)\xi_{\mathbf{r}'}(t')\rangle &= \sigma_m^2 \delta_{\mathbf{r},\mathbf{r}'}\delta(t-t'), \\ \langle \zeta_{\mathbf{r}}(t)\zeta_{\mathbf{r}}(t')\rangle &= \sigma_a^2 \delta_{\mathbf{r},\mathbf{r}'}\delta(t-t'). \end{aligned} \tag{9}$$

Note that for these functions $f(x)$ and $g(x)$ the transitions described are *pure* noise-induced phase transitions, in the sense that they do not exist in the system without noise. The coupling operator $\mathcal{L}$ is a discretized version of the Swift–Hohenberg coupling term $-D(q_0^2+\nabla^2)^2$.[21]

To study the influence of the additive noise, we consider two limiting cases of correlation between additive and multiplicative noise: strong correlation ($\zeta_{\mathbf{r}}=0$ and parameter a is varied), and no correlation ($a=0$ and the intensity of $\zeta_{\mathbf{r}}$ is varied).

Using the generalized Weiss mean field theory (MFT),[5] the conditions of phase transition can be found. Substituting the value of the scalar variable $x_{\mathbf{r}'}$ at the sites coupled to $x_{\mathbf{r}}$ by its special average:

$$\langle x_{\mathbf{r}'}\rangle = \langle x\rangle \cos[\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')], \tag{10}$$

we obtain for $x = x_{\mathbf{r}}$

$$\dot{x} = f(x) + g(x)\xi(t) + D\omega(\mathbf{k})x - D_{\mathrm{eff}}(x-\langle x\rangle) + \zeta(t), \tag{11}$$

where

$$D_{\mathrm{eff}} = \left[\left(\frac{2d}{\Delta^2} - q_0^2\right)^2 + \frac{2d}{\Delta^2} + \omega(\mathbf{k})\right] D \tag{12}$$

and a dispersion relation $\omega(\mathbf{k})=0$ for the most unstable mode, which is only of interest here.[10]

Now the value $\langle x\rangle$ plays the role of the amplitude of the spatial patterns with an effective diffusion coefficient D_{eff}. The steady state solution of the Fokker–Planck equation corresponding to Eq. (10) is written then as follows:

$$w_{st}(x) = \frac{C(\langle x\rangle)}{\sqrt{\sigma_m^2 g^2(x) + \sigma_a^2}} \exp\left(2\int_0^x \frac{f(y) - D_{\mathrm{eff}}(y-\langle x\rangle)}{\sigma_m^2 g^2(y) + \sigma_a^2} dy\right), \tag{13}$$

and $C(\langle x\rangle)$ is the normalization constant.

For the mean field value $\langle x\rangle$ we obtain[21]

$$\langle x\rangle = \int x w_{st}(x,\langle x\rangle) dx. \tag{14}$$

Solving Eq. (14) with parameters D, σ_m^2, and σ_a^2, we obtain a boundary between two phases: a disordered ($|\langle x\rangle|=0$) and an ordered one ($|\langle x\rangle| \neq 0$). The ordered phase corresponds to the appearance of spatially ordered patterns, because its average amplitude becomes nonzero. This happens due to the special form of coupling which includes

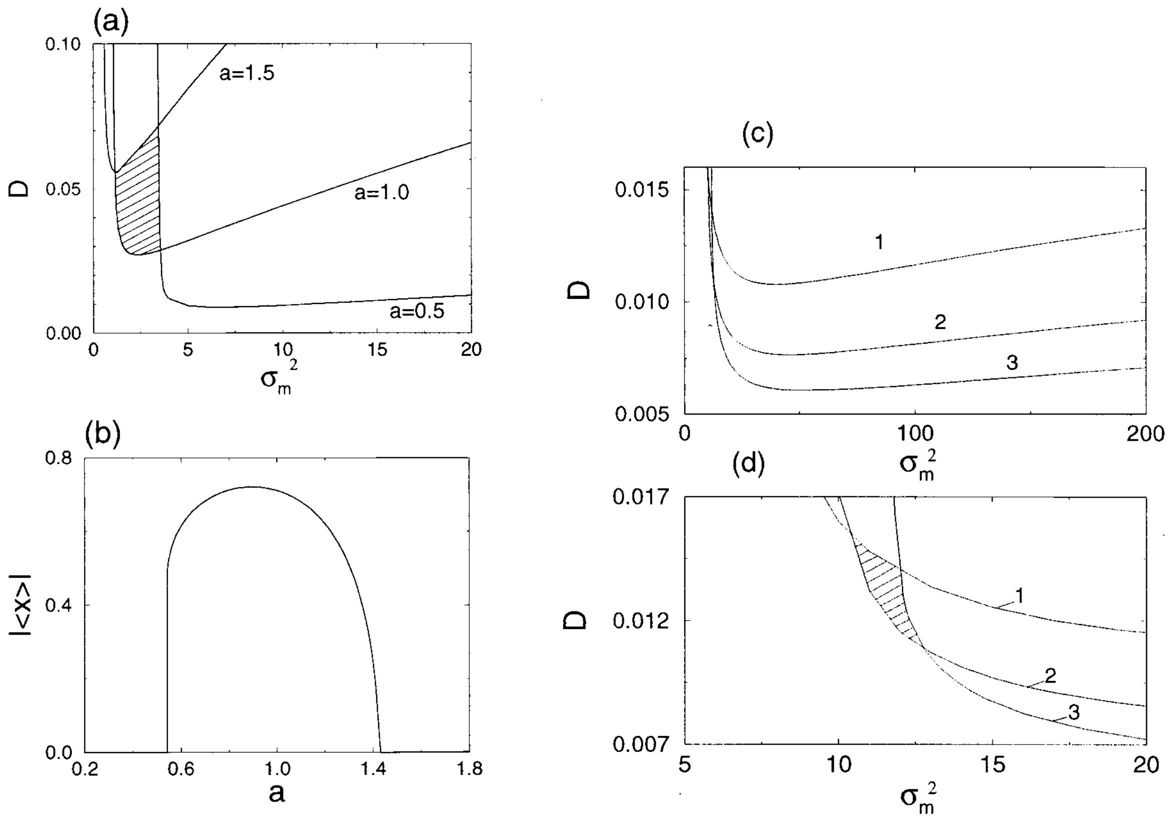

FIG. 3. Additive noise induced phase transition in a nonlinear lattice Eq. (7): predictions of the mean field theory. (a) The boundaries of the transition on the plane (σ_m^2, D) for different values of a Eqs. (8). It is clearly seen that by variation of a a point from the dashed region is a point of the transition induced by additive noise. (b) Dependence of order parameter $|\langle x\rangle|$ if the additive noise intensity is varied. (c) The transition lines for the case when additive and multiplicative noise are independent: $\sigma_a^2 = 1$ (label 1), 0.5 (label 2), and 0.3 (label 3). (d) Large scaled region from the plot in (c).

wave length of these patterns q_0. It is known that in the considered system multiplicative noise induces a phase transition.[10] We focus our attention to the influence of additive noise. The boundary of the phase transition on the plane (σ_m^2, D) is shown in Fig. 3(a), which demonstrates that variation of the intensity of correlated additive noise [the parameter a in Eq. (8)] causes a shift of the transition boundary. The most interesting situation occurs in the dashed region. Here, the increase of the additive noise intensity causes the re-entrant (disorder–order–disorder) phase transition. The corresponding dependence of the order parameter on the parameter a is shown in Fig. 3(b).

For the case of uncorrelated additive noise $(a=0)$, the observed behavior is qualitatively the same [Figs. 3(c) and 3(d)]. Here the transition lines are plotted on the plane (σ_m^2, D) and the intensity σ_a^2 of uncorrelated additive noise is varied. It is evident that again dashed region corresponds to the phase transition. If we take parameters from this dashed region (in both cases of correlation), and change the intensity of additive noise (varying the parameter a or σ_a^2), we observe a formation of patterns and further their destruction (see results of numerical simulations in Fig. 4).

To understand the mechanism behind this transition, it is necessary to note that there is no bistability either in the "usual" potential or in the so-called "stochastic" potential.[4] Nevertheless, using some approximations it can be shown[17,21] that the short-time evolution of the mean field can be described by the "effective" potential, which becomes bistable after a transition. If D, and σ_a^2 vanish, the time evolution of the first moment of a single element is simply given by the drift part in the corresponding Fokker–Planck equation (Stratonovich case)

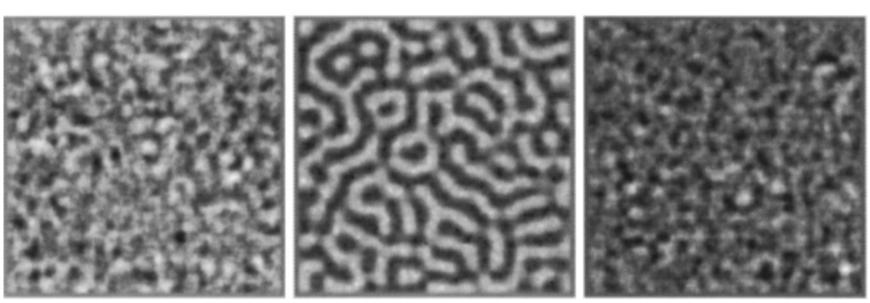

FIG. 4. A formation of spatial patterns induced by additive noise. From left to right the intensity of additive noise is increased $(a=0)$: $\sigma_a^2 = 0.001$, 0.7, and 10 (from left to right). The field in the nonlinear lattice of 128×128 elements is coded from white (minimum) to black (maximum) colors.

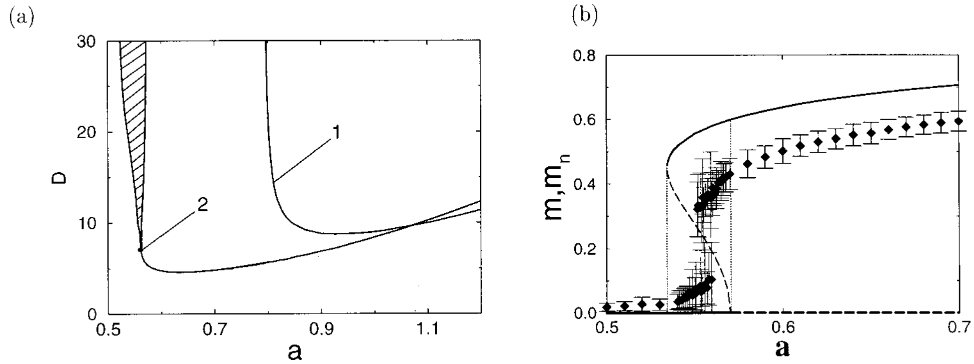

FIG. 5. The nonlinear lattice Eqs. (18): (a) Transition lines on the plane (a,D) for $\sigma_a=0$ and two different intensities of the multiplicative noise (curve 1: $\sigma_m^2=1.6$; curve 2: $\sigma_m^2=3.0$). The dashed region (starting with the dot) corresponds to the coexistence of disordered and ordered phase. (b) The corresponding dependence of the order parameters m, m_n on a for $D=20$, $\sigma_m^2=3.0$, and $\sigma_a^2=0.0$ are plotted by solid lines (MFT predictions) and by diamonds (numerical simulations). The dotted line delimits the coexistence region exhibited by MFT (a region of the hysteresis effect). The unstable state is plotted by the dashed line.

$$\langle \dot{x} \rangle = \langle f(x) \rangle + \frac{\sigma_m^2}{2} \langle g(x) g'(x) \rangle. \tag{15}$$

As it was argued in Ref. 17, the mechanism of the noise-induced transition in coupled systems can be explained by means of a short-time evolution approximation.[30] It means that we start with an initial Dirac δ function, follow it only for a short time, such that fluctuations are small and the probability density is well approximated by a Gaussian. A suppression of fluctuations, performed by coupling, makes this approximation appropriate in our case.[31] The equation for the maximum of the probability, which is also the average value in this approximation $\overline{x}=\langle x \rangle$, takes the following form:

$$\dot{\overline{x}} = f(\overline{x}) + \frac{\sigma_m^2}{2} g(\overline{x}) g'(\overline{x}), \tag{16}$$

which is valid if $f(\langle x \rangle) \gg \langle \delta x^2 \rangle f''(\langle x \rangle)$. For this dynamic, an "effective" potential $U_{\text{eff}}(x)$ can be derived, which has the form

$$U_{\text{eff}}(x) = U_0(x) + U_{\text{noise}} = -\int f(x)\,dx - \frac{\sigma_m^2 g^2(x)}{4}, \tag{17}$$

where $U_0(x)$ is a monostable potential and U_{noise} represents the influence of the multiplicative noise. In the ordered region, this "effective" potential has additional $x=0$ minima that explain the nonzero solutions for the amplitude of spatial patterns.[21]

B. First-order phase transitions

In Ref. 33 a first-order phase transition has been reported, which is induced by multiplicative noise. Now we show that *first-order* nonequilibrium transitions in spatially extended systems can also be induced by additive noise. It is important, that in contrast to second-order transitions, in a first-order transition very tiny fluctuation of the control parameter can lead to a drastic change of the order parameter. The study is performed on a nonlinear lattice of coupled stochastic overdamped oscillators introduced in Ref. 16 and further studied in Refs. 20, 21, 17, and 32. The time evolution of the system is described by the following set of Langevin equations:

$$\dot{x}_i = f(x_i) + g(x_i)\xi_i(t) + \frac{D}{2d} \sum_j (x_j - x_i) + \zeta_i(t), \tag{18}$$

where $x_i(t)$ represents the state of the ith oscillator, and the sum runs over all nearest neighbors of cell i. The strength of the coupling is measured by D, and d is the dimension of the lattice, which has $N=L^d$ elements. The noise terms $\xi_i(t)$ and $\zeta_i(t)$ are the same as defined in Eqs. (9): mutually uncorrelated, Gaussian distributed, with zero mean and white in both space and time. The functions $f(x)$ and $g(x)$ are defined in Eqs. (8).

We study the behavior of this system by means of a standard MFT procedure. Solving the corresponding Eq. (14) with respect to the variable $m=\langle x \rangle$, and w_{st} defined by Eq. (13) with $D_{\text{eff}}=D$, one can set the transition boundaries. In this way obtained order–disorder transition lines are shown in Fig. 5(a). Here we consider only the case when $\sigma_a^2=0$ and the parameter a is varied. Curve 1 separates regions of disorder (below the curve) and order (above the curve) for small multiplicative noise intensity. In this case, the ordered region is characterized by three self-consistent solutions of Eq. (14), one of them unstable $(m=0)$ and the other two stable and symmetrical. These new solutions appear continuously from $m=0$ in the course of the transition. Hence, if we fix the coupling strength, e.g., $D=20$, and increase the intensity of additive noise (the parameter a) a *second-order* phase transition from disorder to order occurs, followed by a re-entrant transition back to disorder, also of second order.

The *first-order* transition can be observed when the multiplicative noise intensity increases. In that case [curve 2 in Fig. 5(a)], a region appears where Eq. (14) has five roots, three of which ($m=0$ and two symmetrical points) are stable. This region is marked dashed in the figure. Thus, for large enough values of D, a region of coexistence appears in the transition between order and disorder. This region is limited by discontinuous transition lines between $m=0$ and a nonzero, finite value of m. Hence, additive noise is seen to induce a *first-order* phase transition in this system for large enough values of the coupling strength and multiplicative noise intensity. The re-entrant transition is again of second order. When the first-order phase transition appears, hysteresis can be expected to occur in the coexistence region (if a certain algorithm is applied[34]). The dependence of the order parameter m on the control parameter a as predicted by MFT is shown in Fig. 5(b) with a solid line. The region of possible hysteresis is bounded by dotted lines.

In order to contrast the analytical results, we have performed simulations of the complete model (18) using the numerical methods described in Refs. 5 and 17. The order parameter m_n is computed as

$$m_n=\left\langle \left| \frac{1}{L^2}\sum_{i=1}^{N} x_i \right| \right\rangle,$$

where $\langle\,\rangle$ denotes time average. Results for a two-dimensional lattice with lateral size $L=32$ are shown with diamonds in Fig. 5(b). Analyzing this figure one can observe that MFT overestimates the size of the coexistence region. This effect, analogous to what was observed for multiplicative-noise induced transitions,[16] can be explained in terms of an "effective potential" derived for the system at short times (see discussion below). For instance, as a increases the system leaves the disordered phase not when this state becomes unstable but earlier, when the potential minima corresponding to the ordered states become much lower than the minimum corresponding to the state $m=0$. It should also be mentioned that the numerical simulations did not show hysteresis, because in the coexistence region the system occupied any of the three possible states, independently of the initial conditions. It can be explained by the small size of the simulated system, which permits jumps between steady states when the system is sufficiently perturbed (e.g., by slightly changing the parameter a).

We have thus seen so far that numerical simulations qualitatively confirm the existence of a first-order phase transition induced by additive noise in this system, as predicted by MFT. We note that the transition occurs in the two limiting cases of correlation between multiplicative and additive noise. We also emphasize that variation of both the multiplicative noise intensity and the coupling strength can change the order of this transition.

Let us now discuss a possible mechanism behind this effect. As pointed out above, the collective behavior of this system can be described by the "effective" potential [see Eq. (17)]. We can trace the behavior of this potential in the presence of multiplicative noise, for the case $\sigma_a^2=0$ and $a\neq 0$. Its evolution for increasing a is shown in Fig. 6. This approach can be clearly seen to successfully explain the mechanism of the first-order transition: first, only the zero state is stable (curve 1), then there is a region where three stable states coexist (curve 2), and finally, the disordered state becomes unstable (curve 3). This approach also explains why a variation of the multiplicative noise intensity influences the order of the transition: for another (lower) σ_m^2 there is no region where ordered and disordered phases simultaneously exist. We emphasize that the "effective" potential is derived only

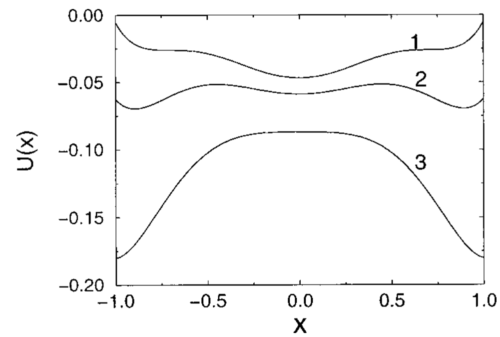

FIG. 6. An "effective" potential for the short-time evolution of m in the lattice Eqs. (18), for $a^2=$:0.25 (curve 1), 0.28 (curve 2), and 0.34 (curve 3). Other parameters are $\sigma_m^2=3.0$ and $\sigma_a^2=0.0$. A coexistence of ordered and disordered phases is observed for the curve 2.

for short-time evolution, and should not be confused with the "stochastic" potential,[4] which for this system remains always monostable. For the other case of correlation between multiplicative and additive noise, in the region of additive noise induced transition, the "effective" potential always has three minima (two symmetric minima are lower than the central one). Sufficiently large (above a threshold of the transition) additive noise causes an escape from zero state and leads to the transition. The value of a critical additive noise intensity for this transition can be estimated by the "effective" potential approach, only by MFT. Here we have considered only a case of strong correlation between multiplicative and additive noise. As described in Ref. 35, if additive noise is independent, it can also induce a first-order phase transition. The level of correlation between additive and multiplicative noise can be considered as an additional parameter in this system, what we leave as an open question here.

In conclusion, we have reported that additive noise can induce a first-order phase transition in a spatially extended system. This transition leads to breaking of symmetry and the creation of a mean field. It should also be mentioned that for another form of coupling, *a la* Swift–Hohenberg as in Sec. IV A, spatial patterns can appear as a result of a first-order phase transition.

V. ADDITIVE NOISE IN DOUBLY STOCHASTIC RESONANCE

Doubly stochastic resonance (DSR)[12] is a synthesis of two basic phenomena: noise-induced phase transition and stochastic resonance (SR).

In the conventional situation SR manifests itself as follows: additive noise optimizes the response of a bistable system to an external periodic force. In addition to this situation, SR has also been found and investigated in a large variety of different class systems: monostable systems,[36] systems with excitable dynamics,[37] noisy non-dynamical systems,[38] systems with sensitive frequency SR dependence,[39] systems without an external force,[8,40] and systems without any explicit threshold.[41] In all these works SR has been observed in the structure, given by the system, and not in the noise-induced structure. In contrast to it, here we address the prob-

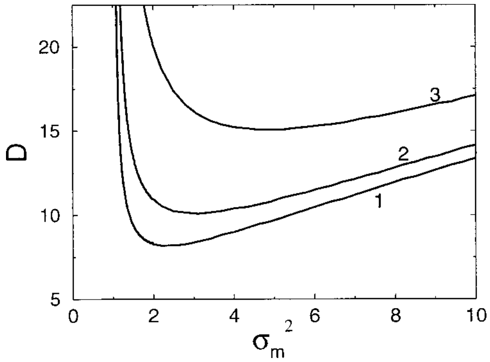

FIG. 7. Transition lines between ordered (inside the curves) and disordered (outside) phase in the lattice Eqs. (19) on the plane $(\sigma_m^2;D)$ for different intensities of the additive noise $\sigma_a^2=0$ (1), 1 (2), and 5 (3). The black dot corresponds to $D=20$, $\sigma_m^2=3$.

lem whether SR can be observed in the bistable structure, which in its own turn is induced by multiplicative noise via phase transition.

We study DSR in the nonlinear lattice of coupled overdamped oscillators Eq. (18), but now under the action of an additional periodic force. Hence, the following set of Langevin equations describes the considered system:

$$\dot{x}_i=f(x_i)+g(x_i)\xi_i(t)+\frac{D}{2d}\sum_j (x_j-x_i)+\zeta_i(t)$$
$$+A\cos(\omega t+\varphi), \qquad (19)$$

where all notations and functions $f(x)$ and $g(x)$ are taken as above. The last term in (19) stands for an external periodic force with amplitude A, frequency ω, and initial phase φ.

Obtained by a standard MFT procedure (see Sec. IV B) transition boundaries between different phases are shown in Fig. 7. In addition to Ref. 16, we show that the influence of additive noise resulted in the shift of transition lines. For $\sigma_a^2=0$ an increase of the multiplicative noise causes a disorder–order phase transition, which is followed by the re-entrant transition to disorder.[16] In the ordered phase the system occupies one of two symmetric possible states with the mean fields $m_1=-m_2\neq 0$, depending on initial conditions (for a visualization of this transition see Fig. 8).

Now let us consider the problem, how the system (19) responds to periodic forcing. We have taken a set of parameters $(\sigma_m^2;D)$ within the region of two coexisting ordered states with a nonzero mean field. In particular, we choose values given by the dot in Fig. 7. For numerical simulations we take a two-dimensional lattice of $L^2=18\times 18$ oscillators, which is simulated numerically[42] with a time step $\Delta t=2.5\times 10^{-4}$ under the action of the harmonic external force. The amplitude of the force A has to be set sufficiently small to avoid hops in the absence of additive noise during the simulation time of a single run which is much larger than the period of the harmonic force.[43] Jumps between $m_1\leftrightarrow m_2$ occur only if additive noise is additionally switched on. Runs are averaged over different initial phases.

Time series of the mean field along the corresponding periodic input signal are plotted in Fig. 9 for three different values of σ_a^2. The current mean field is calculated as $m(t)=(1/L^2)\sum_{i=1}^{N}x_i(t)$. For a small intensity of the additive noise, hops between the two symmetric states m_1 and m_2 are rather seldom and not synchronized to the external force. If we increase the intensity σ_a^2, we achieve a situation when hops occur with the same periodicity as the external force

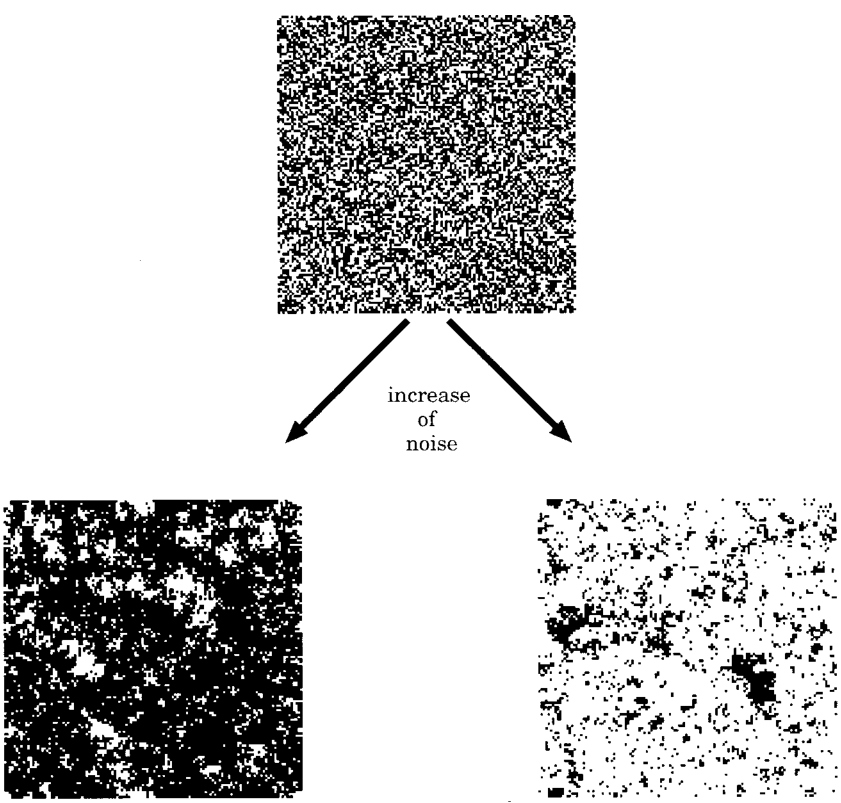

FIG. 8. A symbolic visualization of a phase transition in the model Eqs. (19), which leads to the formation of a mean field. In the disordered phase the mean field is zero due to the random deviation of different elements around zero (up). In the ordered phase, induced by noise, the symmetry is broken and the mean field is either positive (right) or negative (left). The elements in the lattice 128×128 are coded in accordance to its sign: if positive or zero, white; if negative, black.

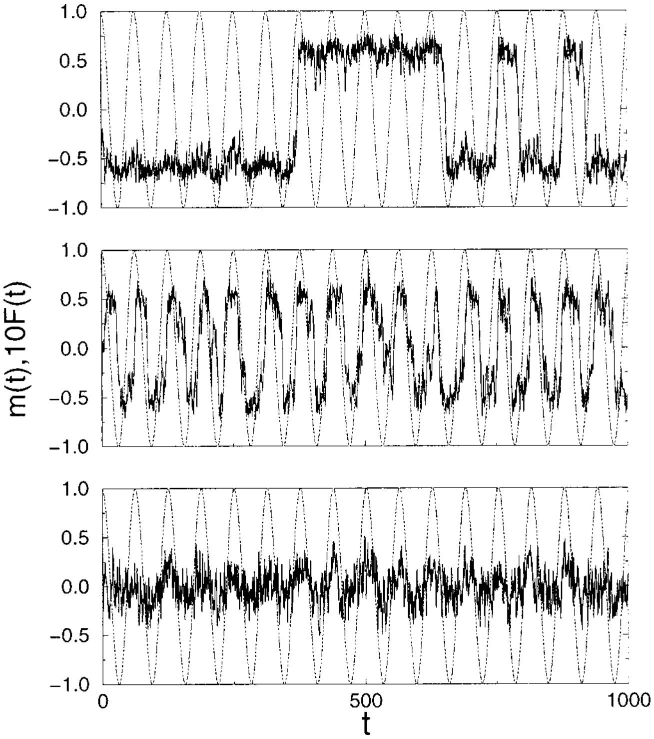

FIG. 9. Doubly stochastic resonance in the lattice (19): a coherent response to periodic driving induced by additive noise. The time evolution of the current mean field (output) and the periodic external force $F(t)$ (input) for different intensities of additive noise (from top to bottom) $\sigma_a^2=0.01$, 1.05, and 5.0. For the optimal value of the additive noise intensity (middle row), hops occur mostly with the period of the external force. The remaining parameters are $A=0.1$, $\omega=0.1$, $D=20$, and $\sigma_m^2=3$.

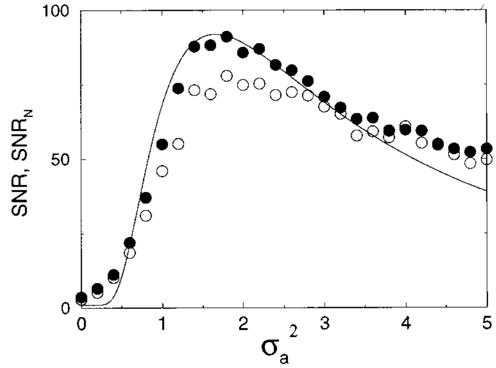

FIG. 10. The dependence of SNR vs the additive noise intensity in the lattice (19). The full output and its two-states approximation are plotted by filled and opaque points, respectively. The solid line shows the analytical estimation SNR_N (22), performed on the base of derivation of the "effective" potential and linear response theory. The parameters are the same as for Fig. 9 and the processing gain $G=0.7$.

and, hence, the mean field follows with high probability the periodic input force. An increase of additive noise provides an optimization of the output of the system which is stochastic resonance. If σ_a^2 is increased further, the order is again destroyed, and hops occur much more frequently than the period of the external force. Note also that for large σ_a^2 the value of the mean field which corresponds to the stable state is becoming smaller. It is caused by the fact that additive noise also influences transition lines.[20] An increase of σ_a^2 results in the reduction of the ordered region (Fig. 7, curves 2 and 3) and decreasing the value $m_1=-m_2$.

Figure 9 illustrates that additive noise is able to optimize the signal processing in the system (19). In order to characterize this SR effect quantitatively, we have calculated signal-to-noise ratio (SNR) by extracting the relevant phase-averaged power spectral density $S(\omega)$ and taking the ratio between its signal part with respect to the noise background.[2] The dependence of SNR on the intensity of the additive noise is shown in the Fig. 10 for the mean field (filled points) and the mean field in a two-state approximation (opaque point). In this two-states approximation we have replaced $m(t)$ by its sign and put approximately $m(t)=+1$ or $m(t)=-1$, respectively. Both curves exhibit the well known bell shaped dependence on σ_a^2 typically for SR.

Next we estimate the SNR analytically, in order to compare it with numerical simulations. If A, D, and σ_a^2 are equal to zero, the dynamics of the system is described by the "effective" potential $U_{\mathrm{eff}}(x)$ [see Eq. (17)]. In the ordered region, inside the transition lines (Fig. 1), the potential $U_{\mathrm{eff}}(x)$ is of the double-well form, e.g., $U(x)_{\mathrm{eff}}=-x^2-0.25x^4+x^6/6$, for given $f(x)$, $g(x)$, and $\sigma_m^2=3$.

From the analytical form of the system's bistable potential, we can solve a conventional SR problem in this potential with an external periodic force of the amplitude A and the frequency ω. Using the well-known approach of a linear response theory,[2,44] we get the following expression for SNR:

$$\mathrm{SNR}_1=\frac{4\pi A^2}{\sigma_a^4}r_k, \tag{20}$$

where r_k is the corresponding Kramers rate[45]

$$r_k=\frac{\sqrt{|U''_{\mathrm{eff}}(x)|_{x=x_{\min}}|U''_{\mathrm{eff}}(x)|_{x=x_{\max}}}}{2\pi}\exp\left(-\frac{2\Delta U_{\mathrm{eff}}}{\sigma_a^2}\right) \tag{21}$$

for surmounting the potential barrier ΔU_{eff}. Using Eqs. (17), (20), and (21) we get analytical estimates for a single element inside the lattice. Further on, rescaling this value by the number N of oscillators in the lattice[46] and taking into account the processing gain G and the bandwidth Δ in the power spectral density,[44] the SNR_N of the mean field of the network of N elements can be obtained

$$\mathrm{SNR}_N=\mathrm{SNR}_1\frac{NG}{\Delta}+1. \tag{22}$$

This dependence is shown in Fig. 10 by the solid line and demonstrates despite the rough approximation a good agreement with the results of the numerical simulations. Nearly exact agreement is found in the location of the maximum as well as for the quantitative values of the SNR ("scalloping loss"[44] has been avoided in simulations by setting the frequency ω to be centered on one of the bins in the spectrum).

In conclusion, we have reported the existence of doubly stochastic resonance, which is resulted from the twofold influence of noise on a nonlinear system. DSR is a combined effect which consists of a noise-induced phase transition and conventional SR. It is important to add, that there are clear distinctions between SR and DSR behavior, because, in contrast to SR, in DSR additive noise does not only help an input/output synchronization, but also changes the properties of the system in the absence of the external force (see Fig. 7). As a consequence, in DSR amplitude of hops is decreased (bistability disappears) for large noise intensities σ_a^2, that is not the case for standard SR (compare Fig. 9 and Fig. 4 from Ref. 2). It means also that a decrease of SNR with the increase of the additive noise intensity can be explained not only by disordered hops induced by large additive noise, but also by the fact that the system loses its bistability. Another distinction is that DSR can be controlled by multiplicative noise, and this control is not possible in a conventional SR. It happens because change of multiplicative noise results in the change of the ''effective'' potential [Eq. (17)], which governs the behavior of the system.

VI. SUMMARY AND OPEN QUESTIONS

We have reported here recent results concerning the influence of additive noise on noise-induced nonequlibrium phase transitions. We have shown that the role of additive noise can be crucial in various aspects: (i) In oscillatory systems, represented by a single oscillator, additive noise is able to induce such NIT, it strongly influences this transition and stabilizes oscillations occurred as a result of this transition. (ii) In spatially extended systems, which are lattices of coupled overdamped oscillators, additive noise can induce first- as well as second-order phase transitions, cause the formation of spatial patterns, and optimize the response of such a system to periodic driving. In the latter case, it is important that the bistability of the collective behavior is supported by multiplicative noise.

Despite these findings there are several open questions and promising directions of future research. Note that the topic of nonequilibrium phase transitions induced by additive noise is rather new. We see three main directions in the study of these transitions.

(1) Theory of noise-induced phase transitions. The phenomena described here are demonstrated by a large variety of models, and the question naturally arises whether these transitions belong to any of the existing universality classes. A discussion about it can be found in Ref. 17 for the transitions which leads to the breaking of symmetry and creation of the mean field. In general, however, this is still an open question as well as a question whether dependencies in the presented models are universal for other models demonstrating these transitions. Another interesting problem is a search of combined effects, as, e.g., a synthesis of white noise driven ratchets and noise-induced nonequilibrium phase transitions,[11] globally synchronized oscillations in subexcitable media[47] or DSR. One should investigate the translation of the transitions discussed into other phenomena, probably systems of coupled excitable elements. It is interesting also to find hidden transitions induced by additive noise in oscillatory systems in the absence of multiplicative noise.[48] Another group of open questions is connected to DSR. We expect that DSR or its modifications can be found not only in the system, described here, but probably in oscillatory systems (see also a case considered in Ref. 19), or systems with a bistable ''stochastic'' potential.[49]

(2) Experimental confirmation of noise-induced transitions predicted by theoretical studies. For the pendulum, modelling a real mechanical object (Sec. II), and the epidemiological model, describing a real experimental data (Sec. III), the connection to the experiment is clear. Concerning spatially extended systems with noise, described in Secs. IV and V, we suggest the following potential experimental implementations. As proposed in Ref. 17, it is worth to reevaluate experiments in physical systems for which noise-induced shifts[15,16] or purely noise-induced transitions may be relevant. Some examples of noise-induced shifts can be mentioned here, such as processes in photosensitive chemical reactions under the influence of fluctuating light intensity,[50,51] in liquid crystals,[52–55] or in the Rayleigh–Bénard instability with a fluctuating temperature at the plates.[56]

We expect also that our theoretical findings will stimulate experimental works to verify DSR in real physical systems (for the first experimental observation of noise-induced bistability see Ref. 57). Appropriate situations can be found in analog[58] or electronic circuits,[59] as well as in systems, which demonstrate noise-induced shifts of the phase transition (see the discussion above). It can be crucial for such experiments, that, in contrast to conventional SR, in DSR the energy of noise is used in a more efficient way: not only for the optimization of the signal processing, but also for the support of the potential barrier to provide this optimization. This can be of a large importance in the communication.

(3) Modelling transitions and irregular oscillations observed in experimental data by stochastic models. As shown in Ref. 14, already known phenomena which have been explained in the frames of a deterministic theory, could also be successfully described by stochastic models. Note that deterministic and noise-induced processes are very difficult to be distinguished in many situations. Moreover, sometimes a noisy excitation looks more justified. It is worth to mention a recently outlined hypothesis that turbulence in nonclosed flows is a result of noise-induced phase transition (Ref. 60 and the experiment in Ref. 61). Also we expect that noise-induced processes may be very important for understanding of complex natural systems studied in neuroscience (e.g., Ref. 62) or such as microseismic oscillations,[63] or phase transitions observed in physiological systems, especially in bimanual movements.[64,65] Despite the fact that up to now these tempo-induced transitions in the production of polyrhythm are explained by deterministic mechanisms in the presence of noise, we expect that models with noise-induced oscillations will also be relevant in this case.

Another open question, closely associated with modelling is the identification of the excitation mechanism by the analysis of irregular time series. This problem is of high importance, because to model a system one should know the physical mechanism of an excitation. At the same time, time series are often the single source of the information about a nonlinear system: ''black box.'' At this point, it is essential to note that classical methods of analysis, such as a spectral analysis or a calculation of a correlation dimension are sometimes unable to distinguish between noise-induced irregular oscillations and chaotic oscillations of the deterministic nature.[13]

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