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Simple electronic circuit model for doubly stochastic resonance

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We have recently reported the phenomenon of doubly stochastic resonance [Phys. Rev. Lett. **85**, 227 (2000)], a synthesis of noise-induced transition and stochastic resonance. The essential feature of this phenomenon is that multiplicative noise induces a bimodality and additive noise causes stochastic resonance behavior in the induced structure. In the present paper we outline possible applications of this effect and design a simple lattice of electronic circuits for the experimental realization of doubly stochastic resonance.

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Investigations of phenomena such as noise-induced phase transitions [1–5], stochastic transport in ratchets [6], or noise-induced pattern formation [7] have shown that the energy of noise, which was usually considered as a nuisance in any communication, can be potentially useful to induce order in nonlinear nonequilibrium systems. One of the most important examples is stochastic resonance (SR) [8,9], which has been found in different engineering [10] and natural systems [11]. In the conventional situation this effect consists of the following: additive noise can optimize the signal processing in a bistable system, i.e., it increases the signal-to-noise ratio in the output if a periodic signal acts upon a system. In addition to this conventional situation, SR has been also found in monostable systems [12], systems with excitable dynamics [13], noisy nondynamical systems [14], systems without an external force [15] (note also coherence resonance [16]), systems without any kind of threshold [17], and systems with transient noise-induced structure [18].

However, the energy of noise can be used much more efficiently: The main point is to use noise not only for a synchronization of output hops across a potential barrier with an external signal, but also for the construction of this barrier. This happens in the effect of doubly stochastic resonance (DSR) [19]. In DSR the influence of noise is twofold: additive noise induces resonancelike behavior in the structure, which has been, in turn, induced by multiplicative noise. DSR occurs in a spatially distributed system of coupled overdamped oscillators and can be considered as a synthesis of two basic phenomena: SR and a noise-induced phase transition [20].

An important question is, How can we observe DSR in experimental systems? We have mentioned in Ref. [19] several appropriate real systems: analog circuits [21], liquid crystals [22], photosensitive chemical reactions [23], Rayleigh-Bénard convection [24], or liquid helium [25]. In the present Rapid Communication we design an electronic circuit for the observation of DSR. The most direct way is the realization through analog circuits, but there are complications due to the complex construction of every unit; hence, it is worth looking for a simpler electronic circuit model that exhibits the DSR property. With this aim we consider an electrical circuit which consists of N coupled elements (i,j). A circuit of one element is shown in Fig. 1. Three ingredients in this circuit are important: the input current, a time-varying resistor (TVR), and a nonlinear resistor. Every element is coupled with its neighbors by the resistor R_c (i.e., by diffusive coupling). The capacitor is shown by C. The nonlinear resistor R_N can be realized with a set of ordinary diodes [26,27], whose characteristic function is a piecewise-linear function

$$i_N = f_1(V) = \begin{cases} G_b V + (G_a - G_b) B_p & \text{if } V \leqslant -B_p \\ G_a V & \text{if } |V| < B_p \\ G_b V - (G_a - G_b) B_p & \text{if } V \geqslant B_p, \end{cases} \quad (1)$$

where i_N is the current through the nonlinear resistor (R_N), V is the voltage across the capacitor (C), and parameters G_a, G_b, and B_p determine the slopes and the breakpoint of the piecewise-linear characteristic curve. Another way to realize the nonlinear resistor is via a third-order polynomial function,

$$i_N = f_2(V) = g_1 V + g_2 V^3.$$

The next important ingredient is a time-varying resistor (TVR) [28,27]. The conductance $G(t)$ of TVRs varies with time. Presently, we consider the case that the function which represents the variation of the TVR is Gaussian δ-correlated in space and time noise, i.e., $G(t) = \xi(t)$, where

$$\langle \xi_i(t) \xi_j(t') \rangle = \sigma_m^2 \delta_{i,j} \delta(t - t').$$

An external action on the circuit is performed by the current input $I(t)$, which is a periodic signal (with amplitude A, frequency ω, and initial phase φ), additively influenced by independent Gaussian noise $\zeta(t)$,

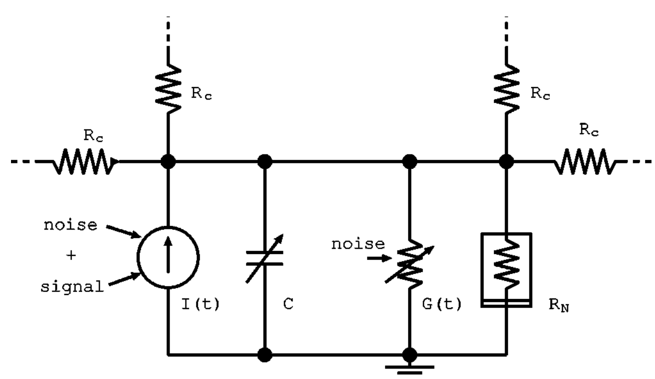

FIG. 1. Electronic circuit of the element (i,j).

$$I(t)=\zeta(t)+A\cos(\omega t+\varphi),$$

where

$$\langle\zeta_i(t)\zeta_j(t')\rangle=\sigma_a^2\delta_{i,j}\delta(t-t').$$

The electronic circuit with respect to the element (i,j) can be described by a set of Kirchoff's equations,

$$C\frac{dV_{i,j}}{dt}=I(t)-G(t)V_{i,j}-f_{1,2}(V_{i,j})+\frac{1}{R_c}(V_{i+1,j}+V_{i-1,j}+V_{i,j+1}+V_{i,j-1}-4V_{i,j}). \quad (2)$$

Hence, the following set of Langevin equations describes the considered system,

$$\frac{dV_{i,j}}{dt}=-f_{1,2}(V_{i,j})+V_{i,j}\xi_{i,j}(t)+\frac{D}{4}(V_{i+1,j}+V_{i-1,j}+V_{i,j+1}+V_{i,j-1}-4V_{i,j})+\zeta_{i,j}(t)+A\cos(\omega t+\varphi), \quad (3)$$

where C is set to unity by normalization of time and D denotes a strength of coupling equal to $4/CR_c$. In the case when f_2 represents the TVR, the model is the time-dependent Ginzburg-Landau equation, which is a standard model to describe phase transitions and critical phenomena in both equilibrium and nonequilibrium situations [3]. It is important that we consider only the situation when the potential of one element is monostable ($G_a=0.5$, $G_b=10$, and $B_p=1$ for f_1; $g_1>0$ and $g_2=1$ for f_2), avoiding the possibility to observe SR without multiplicative noise (The effect of SR in the system, which consists of bistable elements, is well-known and beyond the scope of this paper).

We are interested in the behavior of the mean field $m(t)=(1/N)\Sigma_{i=1}^N\Sigma_{j=1}^N V_{i,j}(t)$ and consider it as an output and the periodic signal as an input of the whole system. SR behavior can be expected if the system is bistable for the chosen set of parameters. Regions of bistability can be determined by means of a standard mean-field theory (MFT) procedure [3]. The mean-field approximation consists of replacing the nearest-neighbor interaction by a global term in the Fokker-Planck equation corresponding to Eq. (3). In this way, we obtain the following steady-state probability distribution w_{st}:

$$w_{\text{st}}(x,m)=\frac{C(m)}{\sqrt{\sigma_m^2g^2(x)+\sigma_a^2}}\times\exp\left(2\int_0^x\frac{f_{1,2}(y)-D(y-m)}{\sigma_m^2g^2(y)+\sigma_a^2}dy\right), \quad (4)$$

where $C(m)$ is a normalization constant and m is a mean field, defined by the equation

$$m=\int_{-\infty}^{\infty}xw_{\text{st}}(x,m)dx. \quad (5)$$

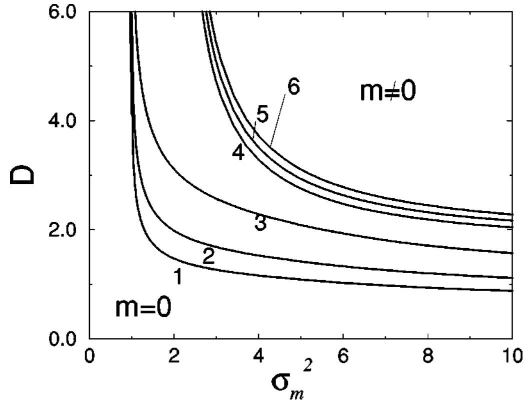

FIG. 2. Transition lines for the equation with function f_1: $\sigma_a^2=0.3$ (label 1), 0.5 (label 2), and 1 (label 3). Also the case with f_2 (the potential of every element is monostable: $g_1>0,g_2=1$); $g_1=1,\sigma_a^2=0.8$ (label 4), 0.9 (label 5), and 1 (label 6).

A self-consistent solution of Eq. (5) determines the mean field and the transition lines between ordered bistable ($m\neq0$) and disordered monostable ($m=0$) phases. Transition boundaries for functions f_1 and f_2 are shown in Fig. 2. Note that bistability is impossible without multiplicative noise and without coupling between elements. Since the SR effect, described below, appears due to the variation of additive noise, it is also important that a change of the additive noise intensity shifts transition boundaries.

Next we estimate the signal-to-ratio (SNR) analytically. Following the short-time evolution approximation, first introduced in [29] and further developed in [30,19], the dynamics of the mean field is governed by an ''effective'' potential $U_{\text{eff}}(x)$, which has the form

$$U_{\text{eff}}(V)=U_0(V)+U_{\text{noise}}=\int f(V)dx-\frac{\sigma_m^2V^2}{4}, \quad (6)$$

where $U_0(V)$ is a monostable potential and U_{noise} represents the influence of the multiplicative noise. Note that this approach is valid only if a suppression of fluctuations, performed by the coupling, is sufficient. It means that the coupling strength should tend to infinity, or actually be large enough. DSR is expected for the regions where this effective potential has a bistable form. To obtain an analytical estimation of SNR for one element we use a standard linear response theory [9,31], yielding

$$SNR_1=\frac{4\pi A^2}{\sigma_\zeta^4}r_k, \quad (7)$$

where r_k is the corresponding Kramers rate [32]

$$r_k=\frac{\sqrt{(|U''_{\text{eff}}(V)|_{V=V_{\min}}|U''_{\text{eff}}(V)|_{V=V_{\max}})}}{2\pi}\exp\left(-\frac{2\Delta U_{\text{eff}}}{\sigma_\zeta^2}\right). \quad (8)$$

Further, we rescale this value by the number N of elements in the circuit [33] and take into account the processing

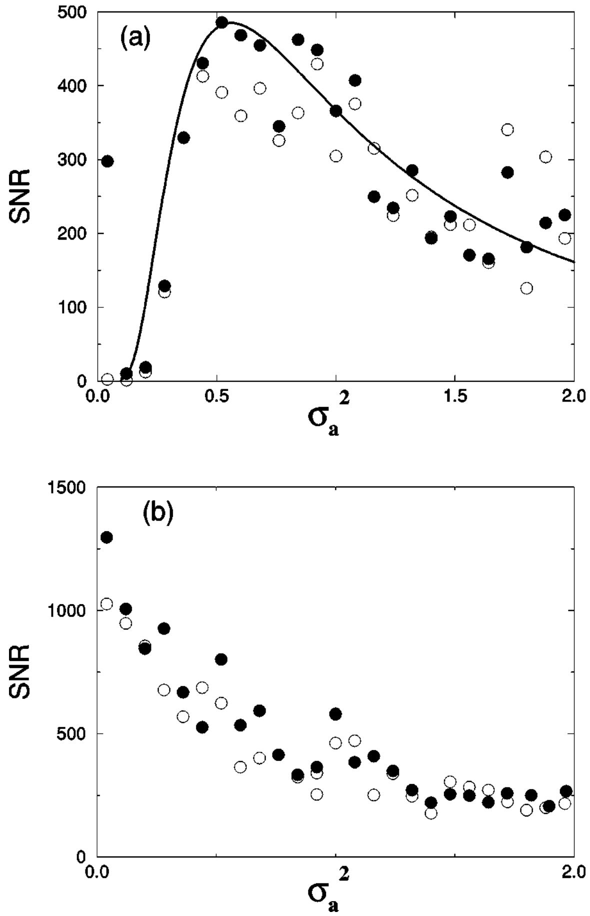

FIG. 3. (a) Numerical SNR (circles) vs analytical estimation (solid line) for the equation with f_1 and $D=3,\sigma_m^2=3$. Numerical results are shown by closed circles for the mean field and open circles for its two-state approximation. The stochastic resonance effect is supported by noise. If we decrease the intensity of multiplicative noise, we do not observe it; e.g., for (b) $D=3,\sigma_m^2=0.5$.

gain G and the bandwidth Δ in the power spectral density [31]. The SNR_N of the mean field of the whole system of N elements is then

$$SNR_N = SNR_1 \frac{NG}{\Delta} + 1. \tag{9}$$

For the parameters, used below for numerical simulations ($\sigma_m^2=3$, $A=0.1$, $N=324$, $G=0.7$, and $\Delta=0.012$), we obtain the analytic estimation of the SNR, shown in Fig. 3(a) by the solid line. Except for the application for electronic circuits, this calculation also shows that DSR can be observed not only in the specific model described in Ref. [19].

In order to verify the results obtained by our rough analytical approximation, we have performed simulations of model (3) using numerical methods described in Ref. [34]. We have taken a set of parameters within the region of two coexisting ordered states with nonzero mean field. As a total system, we take a two-dimensional lattice of 18×18 elements, which was simulated numerically with a time step $\Delta t=2.5\times10^{-4}$. The amplitude of the external signal was set to 0.1, i.e., sufficiently small to avoid hops between two states in the absence of additive noise. To describe the SR effect quantitatively, we have calculated the SNR by extracting the relevant phase-averaged power spectral density $S(\omega)$ and taking the ratio between its signal part with respect to the noise background [9]. The dependence of the SNR on the intensity of the additive noise is shown in Fig. 3(a) for the mean field (closed circles) and the mean field in a two-state approximation (open circles). In this two-state approximation, we have replaced the value of the mean field in time-series by its sign before calculating the power spectral density, using the method of symbolic dynamics [35], standardly used to investigate SR [9]. Both curves demonstrate well-known bell-shaped dependence that is typical for SR. In contrast to two-state approximation, for the mean field, SNR tends to infinity for small values of multiplicative noise intensity (see closed circles for $\sigma_a^2<0.1$). It can be explained by intrawell dynamics in the same way as in the conventional SR [9]. Numerical simulations agree very well with our theoretical estimation despite the very rough approximation via ''effective'' potential (we will study the question, what is the parameters regions of its validity, in a future publication).

Note that this SR effect is created by multiplicative noise, since a bimodality is induced by the combined actions of the multiplicative noise and the coupling. If we decrease only the intensity of multiplicative noise, other parameters fixed, the SR effect is not observed, as is shown in Fig. 3(b). The reason is that in this case our system is not bistable (see Fig. 2). For f_2 the behavior is similar: DSR is observed for $g_1=1,g_2=1,D=5$, $\sigma_m^2=5$, but not for $\sigma_m^2=3,D=5$. For the experimental setup a minimal number of elements, which is neccessary for DSR observation, can be important. Reduction of the element number in this system leads to the fact that a system can spontaneously (even in the absence of forcing) perform a hop between two states. These jumps hide the DSR effect, since they destroy a coherence between input and output. For the system size 18×18, considered here, such jumps are rather seldom [36] and do not hinder DSR. Our calculations have shown that a size 10×10 is still satisfactory, whereas further decrease of the element number will destroy the effect.

In conclusion, we have proposed a rather simple electronic circuit implementation of the DSR effect in order to encourage observers to perform this or a similar experiment. It is important to add that in spite of the fact that the DSR can be interpreted as some modification of SR, there are several important distinctions between DSR and conventional SR. First, a potential barrier is supported by multiplicative noise; it means that DSR is very efficient from the energetic viewpoint. Another consequence is that this SR effect can be controlled by a variation of multiplicative noise intensity. Second, in contrast to SR, the amplitude of hops is changed if we change the intensity of additive noise (similar to Fig. 3 from [19]). This is explained by the fact that an increase of additive noise influences the transition lines (see Fig. 2) and decreases the mean field, which corresponds to a stable position in the absence of the external force.

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