Influence of additive noise on transitions in nonlinear systems

P. S. Landa,[1] A. A. Zaikin,[2] V. G. Ushakov,[1] and J. Kurths[2]
[1]Lomonosov Moscow State University, 119899 Moscow, Russia
[2]Institute of Physics, University of Potsdam, Am Neuen Palais 10, 14469 Potsdam, Germany
(Received 30 August 1999; revised manuscript received 12 November 1999)

The effect of additive noise on transitions in nonlinear systems far from equilibrium is studied. It is shown that additive noise in itself can induce a hidden phase transition, which is similar to the transition induced by multiplicative noise in a nonlinear oscillator [P. Landa and A. Zaikin, Phys. Rev. E **54**, 3535 (1996)]. Investigation of different nonlinear models that demonstrate phase transitions induced by multiplicative noise shows that the influence of additive noise upon such phase transitions can be crucial: additive noise can either blur such a transition or stabilize noise-induced oscillations.

PACS number(s): 05.40.−a, 05.70.Fh

I. INTRODUCTION

Noise-induced transitions occupy an important place among phenomena that demonstrate a strong influence of weak noise on the behavior of a system [2]; e.g., *stochastic resonance* [3–5], *noise-induced transport* [6], *coherence resonance* [7] or *noise-induced pattern formation* [8]. Intensive investigations of recent years have shown that noise-induced phase transitions can manifest themselves in the appearance of new extrema in the system probability distribution [9,10], in the creation of a mean field [11–13], and in the excitation of oscillations [1,14,15]. The last two types of transitions [16] have been termed nonequilibrium noise-induced phase transitions [17,1].

In these and other works multiplicative noise is perceived to be responsible for the transitions. However, as has been recently shown in [18–20], additive noise plays a crucial role in these transitions. Hence, studying the influence of additive noise is of great importance. In this paper we study several major aspects of the influence of additive noise by consideration of typical models in which a transition leads to noise-induced oscillations.

First, we study a transition induced by multiplicative noise in the presence of additive noise. We investigate such a transition theoretically and numerically in a pendulum with randomly vibrating suspension axis. In this model the additive noise blurs the transition induced by multiplicative noise. The pendulum is a key model for understanding another effect: a hidden phase transition induced purely by additive noise. We demonstrate it for an oscillator with quadratic nonlinearity and random force by showing that autoparametrical excitation occurs due to the additive noise and quadratic nonlinearity. At the same time the presence of additive noise makes this transition hidden. The mechanism of this transition is similar to subharmonic resonance [14]. Another mechanism, combination resonance, can also be associated with a phase transition induced by additive noise. This mechanism is illustrated by an electromechanical vibrator energized from a source of sufficiently high-frequency random current in place of a periodic source [21,14]. The combination resonance is caused by nonlinear interaction of random oscillations of the source current and the oscillations induced in the high-frequency subsystem. Then we consider a standard epidemiological model [22–24] with a random action and show that this action can be split into additive and multiplicative parts. In contrast to the pendulum, here the transition can be induced by both additive and multiplicative noise. The mechanisms are likely to be the same as in the oscillator with quadratic nonlinearity and in the pendulum, respectively. The combined action of additive and multiplicative noise in this system extends the range of the parameters where noise-induced oscillations are stable, so we interpret this phenomenon as stabilization of noise-induced oscillations by additive noise.

The organization of the paper is as follows. In Sec. II we consider a pendulum with multiplicative and additive noise, which demonstrates a phase transition induced by multiplicative and influenced by additive noise. In Sec. III systems with additive noise alone are considered: an oscillator with quadratic nonlinearity and an electromechanical vibrator. Section IV is devoted to the study of transitions induced by both additive and multiplicative noise and of the stabilizing influence of additive noise in an epidemiological model. In Sec. V we summarize the results obtained.

II. NOISE-INDUCED PHASE TRANSITIONS IN THE PRESENCE OF ADDITIVE NOISE

First, we study the problem of excitation of a nonlinear oscillator under parametric and forcing random actions. We give an approximate analytical solution of this problem to reveal the influence of additive noise on a phase transition induced by multiplicative noise in a pendulum with a randomly vibrated suspension axis. In the absence of additive noise such a transition has been considered in [1,25]. It should be noted that the additive constituent of noise appears by itself if the vibration of the pendulum's suspension axis occurs in a certain direction making a nonzero angle with the vertical [18].

In the presence of additive noise the equation of motion for this system can be written as

$$\ddot{\varphi}+2\beta(1+\alpha\dot{\varphi}^2)\dot{\varphi}+\omega_0^2[1+\xi_1(t)]\sin\varphi=\omega_0^2\xi_2(t), \quad (1)$$

where φ is the pendulum's angular deviation from the equilibrium position, ω_0 is the natural frequency of a small free pendulum's oscillations, β is the damping factor, α is the coefficient of nonlinear friction, and $\xi_1(t)$ and $\xi_2(t)$ are comparatively broadband random processes with zero mean values. We assume that the suspension axis vibration is moderately small in amplitude, i.e., the pendulum oscillations can be considered small enough for φ to be substituted in place of $\sin\varphi$ in Eq. (1).

An approximate analytical solution of this problem can be obtained from the assumptions that $\beta/\omega_0 \sim \epsilon$, $\xi_1(t) \sim \sqrt{\epsilon}$, and $\xi_2(t) \sim \sqrt{\epsilon}$, where ϵ is a certain small parameter which should be put equal to unity in the final results. Equation (1) can then be solved by the Krylov-Bogolyubov method; to do this we set $\varphi = A(t)\cos\psi(t) + \epsilon u_1 + \cdots$, where $\psi(t) = \omega_0 t + \phi(t)$,

$$\dot{A} = \epsilon f_1 + \cdots, \quad \dot{\phi} = \epsilon F_1 + \cdots, \tag{2}$$

and $u_1, \ldots, f_1, \ldots, F_1, \ldots$ are unknown functions. By using the Krylov-Bogolyuov technique for stochastic equations (see [26]), we find expressions for the unknown functions f_1 and F_1. Substituting these expressions into Eqs. (2) we obtain

$$\dot{A} = -\beta(1 + \tfrac{3}{4}\alpha\omega_0^2 A^2)A + \omega_0 \overline{g_1(A, \omega(t), \xi_1(t), \xi_2(t))}, \tag{3}$$

$$\dot{\phi} = \omega_0 \overline{g_2(A, \psi(t), \xi_1(t), \xi_2(t))}, \tag{4}$$

where

$$g_1(A,\phi,t) = \frac{A}{2}\xi_1(t)\sin 2\psi(t) - \xi_2(t)\sin\psi(t),$$

$$g_2(A,\phi,t) = \xi_1(t)\cos^2\psi(t) - \frac{1}{A}\xi_2(t)\cos\psi(t).$$

The bar over an expression denotes averaging over time.

As follows from [26], the Fokker-Planck equation associated with Eqs. (3) and (4) is

$$\frac{\partial w(A,\phi,t)}{\partial t} = -\frac{\partial}{\partial A}\{[-\beta(1 + \tfrac{3}{4}\alpha\omega_0^2 A^2)A + \omega_0^2 R_1] \times w(A,\phi,t)\} - \omega_0^2 R_2 \frac{\partial w(A,\phi,t)}{\partial\phi} + \frac{\omega_0^2}{2} \times \left\{\frac{\partial^2}{\partial A^2}\left[\left(\frac{K_{11}}{4}A^2 + K_{12}\right)w(A,\phi,t)\right] + \left(K_{21} + \frac{K_{22}}{A^2}\right)\frac{\partial^2 w(A,\phi,t)}{\partial\phi^2}\right\}, \tag{5}$$

where

$$R_1 = \int_{-\infty}^{0}\left(\left\langle\overline{\frac{\partial g_1(A,\phi,t)}{\partial A} g_1(A,\phi,t+\tau)}\right\rangle + \left\langle\overline{\frac{\partial g_1(A,\phi,t)}{\partial\phi} g_2(A,\phi,t+\tau)}\right\rangle\right)d\tau, \tag{6}$$

$$R_2 = \int_{-\infty}^{0}\left(\left\langle\overline{\frac{\partial g_2(A,\phi,t)}{\partial A} g_1(A,\phi,t+\tau)}\right\rangle + \left\langle\overline{\frac{\partial g_2(A,\phi,t)}{\partial\phi} g_2(A,\phi,t+\tau)}\right\rangle\right)d\tau, \tag{7}$$

(the angular brackets denoting averaging over the statistical ensemble),

$$K_{11} = \tfrac{1}{2}\kappa_{\xi_1}(2\omega_0), \quad K_{12} = \tfrac{1}{2}\kappa_{\xi_2}(\omega_0), \tag{8}$$

$$K_{21} = \tfrac{1}{4}[\kappa_{\xi_1}(0) + \tfrac{1}{2}\kappa_{\xi_1}(2\omega_0)],$$

$$K_{22} = \tfrac{1}{4}[\kappa_{\xi_2}(0) + \tfrac{1}{2}\kappa_{\xi_2}(\omega_0)], \tag{9}$$

and

$$\kappa_\xi(\omega) = \int_{-\infty}^{\infty}\langle\xi(t)\xi(t+\tau)\rangle\cos\omega\tau\, d\tau$$

is the power spectrum density of the process $\xi(t)$ at the frequency ω.

Let us now calculate the integrals (6) and (7), taking into account the expressions for g_1 and g_2. As a result, we obtain

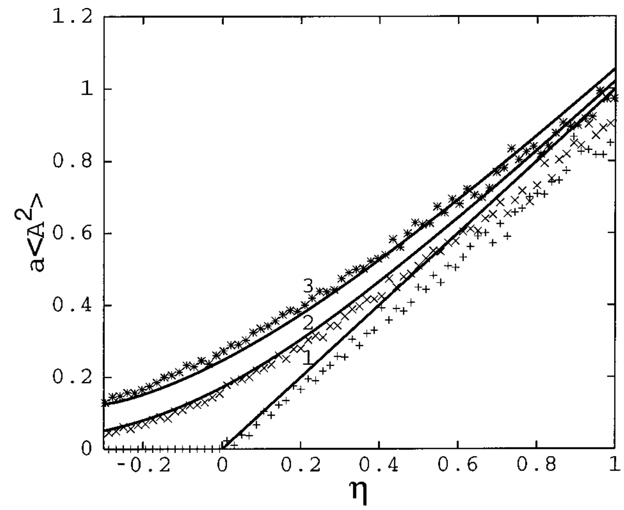

FIG. 1. The influence of additive noise on a noise-induced phase transition in a pendulum with randomly vibrated suspension axis. The dependence of the value $a\langle A^2\rangle$, which is proportional to the mean amplitude squared, on η without additive noise $q_0=0$ and with additive noise $q_0=0.005$ and 0.02 for curves 1–3 respectively. Theoretical (solid lines) and numerical results (symbols). In the presence of additive noise the dependence is smooth. The remaining parameters are $\beta=0.1$, $\alpha=100$, and $\omega_0=1$.

$$R_1=\frac{3A}{8}\int_{-\infty}^{0}\langle\xi_1(t)\xi_1(t+\tau)\rangle\cos 2\omega_0\tau d\tau+\frac{1}{2A}\int_{-\infty}^{0}\langle\xi_2(t)\xi_2(t+\tau)\rangle\cos\omega_0\tau d\tau=\frac{3K_{11}}{8}A+\frac{K_{12}}{2A}, \tag{10}$$

$$R_2=\frac{1}{4}\int_{-\infty}^{0}\langle\xi_1(t)\xi_1(t+\tau)\rangle\sin 2\omega_0\tau d\tau-\frac{1}{A^2}\int_{-\infty}^{0}\langle\xi_2(t)\xi_2(t+\tau)\rangle\sin\omega_0\tau d\tau. \tag{11}$$

The value of R_2 depends on the characteristics of the random processes $\xi_1(t)$ and $\xi_2(t)$: if they are white noises then $R_2=0$; but if, for example, $\xi_2(t)$ is white noise and $\xi_1(t)$ has a finite correlation time and its power spectrum density is

$$\kappa_{\xi_1}(\omega)=\frac{a_1^2\kappa_{\xi_1}(2\omega_0)}{(\omega-2\omega_0)^2+a_1^2},$$

then

$$R_2=-\frac{a_1\omega_0\kappa_{\xi_1}(2\omega_0)}{4(16\omega_0^2+a_1^2)}.$$

It should be noted that in this case R_2 is negative, which results in a decrease of the mean oscillation frequency with increasing noise intensity. The Langevin equations which can be related to the Fokker-Planck equation (5) in view of Eqs. (10) and (11) are presented in Appendix A.

First we consider *the case when additive noise is absent*, i.e., $\kappa_{\xi_2}\equiv 0$. In this case the steady-state solution of Eq. (5), satisfying the condition of zero probability flux, is

$$w(A,\phi)=\frac{C}{2\pi A^2}\exp\left[\frac{3}{1+\eta}\left(\eta\ln A-\frac{aA^2}{2}\right)\right], \tag{12}$$

where $a=3\alpha\omega_0^2/4$ is the nonlinear parameter and $\eta=3\omega_0^2K_{11}/8\beta-1$. The constant C is determined from the normalization condition

$$\int_0^{2\pi}\int_0^{\infty}w(A,\phi)AdAd\phi=1.$$

Upon integrating Eq. (12) over ϕ, we find the expression for the probability density w of the oscillation amplitude:

$$w(A)=CA^{(2\eta-1)/(1+\eta)}\exp\left(-\frac{3aA^2}{2(1+\eta)}\right). \tag{13}$$

From the normalization condition we get

$$C=2\times\begin{cases}\left(\dfrac{3a}{2(1+\eta)}\right)^{3\eta/2(1+\eta)}\dfrac{1}{\Gamma(3\eta/2(1+\eta))} & \text{for}\quad \eta\geqslant 0\\ 0 \quad \text{for}\quad \eta\leqslant 0.\end{cases} \tag{14}$$

Hence,

$$w(A)=2\times\begin{cases}\left(\dfrac{3a}{2(1+\eta)}\right)^{3\eta/2(1+\eta)}\dfrac{A^{(2\eta-1)/(1+\eta)}}{\Gamma(3\eta/2(1+\eta))}\exp\left(-\dfrac{3aA^2}{2(1+\eta)}\right) & \text{for}\quad\eta\geqslant 0\\ \delta(A)\quad \text{for}\quad \eta\leqslant 0.\end{cases} \tag{15}$$

The fact that for $\eta\leqslant 0$ the probability density of the amplitude turns out to be a δ function is associated with the absence of additive noise (see below).

Using Eq. (15), we can determine $\langle A\rangle$ and $\langle A^2\rangle$:

$$\langle A\rangle=\begin{cases}\sqrt{\dfrac{3}{2a(1+\eta)}}\dfrac{\Gamma((4\eta+1)/2(1+\eta))}{\Gamma(3\eta/2(1+\eta)+1)}\eta & \text{for}\quad\eta\geqslant 0\\ 0\quad\text{for}\quad\eta\leqslant 0,\end{cases} \tag{16}$$

$$\langle A^2\rangle=\begin{cases}\dfrac{\eta}{a} & \text{for}\quad\eta\geqslant 0\\ 0 & \text{for}\quad\eta\leqslant 0.\end{cases} \tag{17}$$

Therefore, it is evident that for $\eta>0$ the parametric excitation of pendulum oscillations occurs under the influence of multiplicative noise. This manifests itself in the fact that the mean values of the amplitude and of the amplitude squared become nonzero (Fig. 1, curve 1). This parametric excitation implies a transition of the system to a new state, which can be treated as a phase transition. The condition $\eta=0$ is the threshold for the onset of this phase transition. It follows that, in the absence of additive noise, the critical value of the multiplicative noise intensity is

$$\kappa_\xi^{cr}(2\omega_0)\equiv\kappa_{cr}=\frac{16\beta}{3\omega_0^2}. \tag{18}$$

Hence, the parameter η characterizes the extent to which the intensity of the multiplicative noise component exceeds its critical value.

It should be noted that, for $\eta>0$, the steady state $A=0$ loses its stability and the state $A\neq 0$ becomes stable. At the same time, Eq. (15) implies that the probability density of A^2 is monotonically decreasing with increasing A^2 for any value of $\eta>0$. Hence, in contrast to the transitions considered in [9], the appearance of a new stable state need not be accompanied by the appearance of a new maximum in the system probability distribution [see Fig. 2(a)].

Now let us consider *the case when the intensity of additive noise is not equal to zero*. The steady-state solution of Eq. (5), satisfying the condition of zero probability flux, is conveniently written as

$$w(A,\phi)=\frac{Ca}{2\pi(aA^2+q)}\exp\left[\int\frac{3(\eta-aA^2)aA^2+q}{(1+\eta)(aA^2+q)A}\,dA\right], \tag{19}$$

where $q=4aK_{12}/K_{11}$ characterizes the ratio between the intensities of additive and multiplicative noise.

Following the calculations presented in Appendix B, we get an expression for

$$a\langle A^2\rangle\approx(1+\eta)\left\{\frac{4\mu}{3}\Gamma(2\mu)\Gamma(\tfrac{3}{2}-2\mu)(1+2\mu)\left(2(1-2\mu)+(5-4\mu)\frac{3q}{2(1+\eta)}\right)-\frac{3q}{2(1+\eta)}\right.$$
$$\left.\times\left[\sqrt{\pi}\Gamma(-2\mu)(1-2\mu)\left(\frac{3q}{2(1+\eta)}\right)^{2\mu}+2\Gamma(2\mu)\Gamma(\tfrac{3}{2}-2\mu)(1+2\mu)\right]\right\}\left[\frac{\sqrt{\pi}}{2}\Gamma(-2\mu)(1-2\mu)\left(\frac{3q}{2(1+\eta)}\right)^{2\mu}\right.$$
$$\left.\times\left(2(1+2\mu)+\frac{9q}{2(1+\eta)}\right)+\Gamma(2\mu)\Gamma(\tfrac{3}{2}-2\mu)(1+2\mu)\left(2(1-2\mu)+\frac{3(3-4\mu)q}{2(1+\eta)}\right)\right]^{-1}, \tag{20}$$

where $\mu=3(\eta+q)/4(1+\eta)$. Note that, similarly to the case without additive noise, after a transition no additional maxima appear in the system probability distribution and the shape of this distribution is not qualitatively changed [Fig. 2(b)].

Next we compare these analytical results with numerical simulations. The corresponding dependence of $a\langle A^2\rangle$ on η for different values of the parameter q_0 is illustrated in Fig. 1. We see that additive noise of small intensity results in a smoothing of the dependence of the mean oscillation amplitude squared on the multiplicative noise intensity: it becomes without the break inherent in a phase transition induced by only multiplicative noise. If we increase the additive noise intensity, the transition becomes less detectable (Fig. 1, curve 3).

In a numerical experiment it is more convenient to calculate the variance of the corresponding variable instead of the mean amplitude squared. It is evident that the dependencies of these values on the noise intensity should be similar. Indeed, in the case when the amplitude A is a slowly changing function, the variance is equal to $\langle A^2\rangle/2$. The dependencies of $a\langle A^2\rangle$ on η found by numerical simulation of Eq. (1) for both the presence of additive noise and its absence are shown also in Fig. 1. We find that near the threshold the simulations match the analytical results very well and that the dependencies for $q=0$ can be approximated by a straight line intersecting the abscissa at $\eta=0$. With an increase of η, the growth rate of the variance in numerical simulations is smaller than in the analytical results. This can be explained by the fact that the Krylov-Bogolyubov method is valid only near a threshold.

III. PHASE TRANSITIONS INDUCED BY ADDITIVE NOISE

A. Oscillator with quadratic nonlinearity

In this section we show that the mechanism of the noise-induced phase transition may exist also in an oscillatory system with additive noise only. For this we consider an oscillator with a quadratic nonlinearity and additive random force.

The oscillator under consideration can be described by

$$\ddot{x}+2\beta\dot{x}+\omega_0^2(1+x+\gamma x^2)x=\omega_0^2 b\xi(t), \tag{21}$$

where the friction factor β is assumed to be sufficiently small in comparison with the natural frequency ω_0, $\xi(t)$ is an external force, which is a sufficiently broadband random process with zero mean value, the parameter b is responsible for the noise intensity, and the term γx^3 is introduced to avoid the solution going to infinity [caused by the presence of an unstable singular point of Eq. (21) for $\gamma<0.25$].

At this point it is necessary to note that direct use of the Fokker-Planck equation [26] and its stationary solution does not show that the system probability distribution for variables $(x,\dot{x})$ is qualitatively changed with increase of the noise intensity. However, as we learned from the example in the previous section, the transition can take place despite the facts that there is no noise-induced maximum in the system probability distribution (see Fig. 2) and that the transition is not observable in the dependence of variance on noise intensity (see Fig. 1, curve 3). Obviously, the presence of moderately strong additive noise makes every transition hidden and undetectable. Nevertheless, the mechanism of the noise-induced transition is present in the model and, therefore, we call this phenomenon a *hidden phase transition induced by additive noise*.

To demonstrate the physical mechanism that is responsible for the hidden noise-induced phase transition, we will use the same procedure as for the calculation of subharmonic resonances [14]. First, we decompose x into

$$x(t)=y(t)+\chi(t), \tag{22}$$

where $\chi(t)$ is a random process satisfying the equation

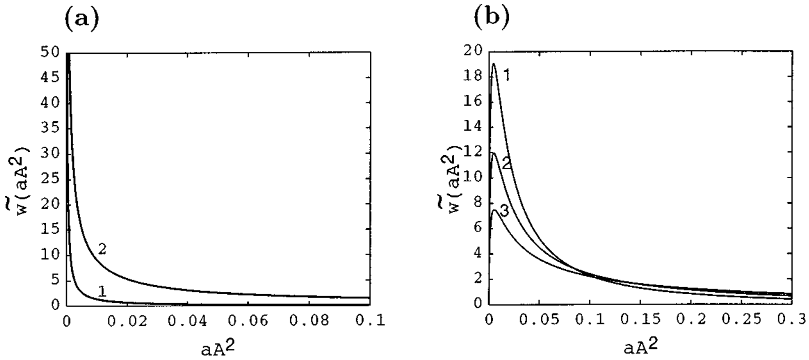

FIG. 2. The system probability distribution for a pendulum. (a) The case without additive noise. The probability distribution $\tilde{w}(aA^2)=w(A)/2aA$ for $\eta=0.01$ (curve 1) and $\eta=0.2$ (curve 2). (b) The case with additive noise. The dependence of $\tilde{w}(aA^2)=w(A)/2aA$ for $q=0.01/(1+\eta)$ and $\eta=-0.2$, 0, and 0.2 for curves 1–3, respectively.

$$\ddot{\chi}+2\beta\dot{\chi}+\omega_0^2\chi=\omega_0^2 b\xi(t). \tag{23}$$

Now we will show that the system described by the variable y undergoes a noise-induced transition. Substituting Eq. (22) into Eq. (21) and taking into account Eq. (23), we get the equation for the variable y,

$$\ddot{y}+2\beta\dot{y}+\omega_0^2\{1+y+\xi_2(t)+\gamma y[y+3\chi(t)]\}y=\omega_0^2\xi_1(t), \tag{24}$$

where $\xi_1(t)=-\chi^2(t)[1+\gamma\chi(t)]$ is additive noise and $\xi_2(t)=\chi(t)[2+3\gamma\chi(t)]$ is multiplicative noise. Comparing Eq. (24) with Eq. (1), we find that these equations are similar. In the absence of additive noise $\xi_1(t)$, Eq. (24) is similar also to Eq. (1) in [1], except that the role of the random process $\xi(t)$ is played by the noise $\xi_2(t)$. In the previous section we have shown analytically and numerically that in the oscillator described by such an equation multiplicative noise causes a phase transition. Hence, the noise $\xi_2(t)$ is responsible for the phase transition, whereas, as will be seen from subsequent results, the additive noise makes the transition hidden.

An approximate analytical analysis of Eq. (24), in view of Eq. (23), is possible in the specific case when the random force in Eq. (23) is nonresonant. Owing to this, $\chi(t)$ is sufficiently small, and we can ignore in Eq. (24) both $\xi_1(t)$ and $3\gamma\chi^2 y$. As a result we obtain the following approximate equation for y:

$$\ddot{y}+2\beta\dot{y}+\omega_0^2[1+y+2\chi+\gamma y(y+3\chi)]y=0. \tag{25}$$

Putting $y=A(t)\cos\psi(t)+\cdots$, where $\psi(t)=\omega_0 t+\phi(t)$, and using now the Krylov-Bogolyubov method for stochastic equations, we obtain the following truncated equations for $A(t)$ and $\phi(t)$:

$$\dot{A}=(\eta-aA^2)A+\omega_0\zeta_1(t), \quad \dot{\phi}=M_1+\omega_0\zeta_2(t), \tag{26}$$

where

$$\eta=\frac{3\omega_0^2K_1}{2\beta}-1, \quad a=\frac{3\gamma}{4}\left(1-\frac{15\gamma\omega_0^2}{8\beta}\right)(K_2+K_3),$$

$$M_1=\int_{-\infty}^{0}\langle\chi(t)\chi(t+\tau)\rangle\left(\sin 2\omega_0\tau+\frac{9\gamma^2A^3}{4}\times(3\sin\omega_0\tau+\sin 3\omega_0\tau)\right)d\tau,$$

$\zeta_1(t)$ and $\zeta_2(t)$ are white noises with intensities

$$N_1=\left(K_1+\frac{9\gamma^2A^2}{16}(K_2+K_3)\right)A^2$$

and

$$N_2=2K_0+K_1+\frac{9\gamma^2A^2}{16}(K_2+K_3),$$

respectively, $K_1=\kappa_\chi(2\omega_0)/2$, $K_2=\kappa_\chi(\omega_0)/2$, $K_3=\kappa_\chi(3\omega_0)/2$, $K_0=\kappa_\chi(0)/2$, and $\kappa_\chi(\omega)$ is the spectral density of the random process $\chi(t)$ at the frequency ω.

Solving the Fokker-Planck equation associated with Eqs. (26), we get the probability density $w(A)$:

$$w(A)=CA^{(2\eta-1)/(1+\eta)}(1+rA^2)^{-[(2+5\eta)r+3a]/2r(1+\eta)}, \tag{27}$$

where

$$r=\frac{9\gamma^2}{16}\frac{K_2+K_3}{K_1}.$$

From the normalization condition we find

$$C=2\times\begin{cases}\dfrac{a+r\eta}{a}r^{3\eta/2(1+\eta)}\dfrac{\Gamma(3(a+r\eta)/2r(1+\eta))}{\Gamma(3\eta/2(1+\eta))\Gamma(3a/2r(1+\eta))} & \text{for } \eta\geqslant 0\\ 0 \quad \text{for } \eta\leqslant 0.\end{cases} \tag{28}$$

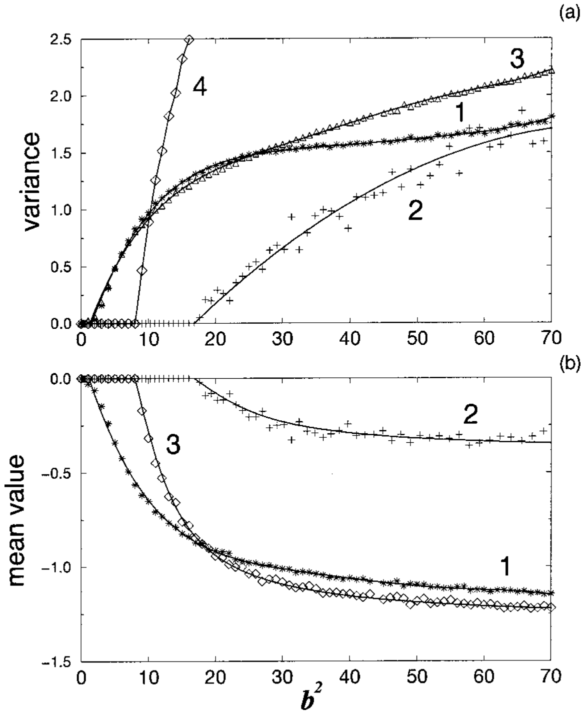

FIG. 3. Dependencies of the first moments of the simulated solutions on b^2 for $\gamma=0.251$, $\omega_0=1$, $\beta=0.1$. (a) variances σ_y^2 (curve 1), $\sigma_{y_r}^2$ (curve 2), σ_x^2 (curve 3), and $\sigma_{y_{rr}}^2$ (curve 4); (b) mean value $\langle y\rangle$ (curve 1), $\langle y_r\rangle$ (curve 2), and $\langle y_{rr}\rangle$ (curve 3) for the same value of γ.

It follows from here that the probability density of the amplitude turns out to be a δ function for $\eta\leqslant 0$, as for the pendulum considered in [1].

Using Eqs. (27) and (28) we calculate $\langle A^2\rangle$ ($\langle\cdot\rangle$ denotes the statistical average):

$$\langle A^2\rangle=\begin{cases}\dfrac{3\eta}{3a+r(2+5\eta)}=\dfrac{4\eta}{3\gamma-4r} & \text{for} \quad \eta\geqslant 0\\ 0 \quad \text{for} \quad \eta\leqslant 0.\end{cases} \tag{29}$$

Note that the solution found is valid only for $3\gamma(K_2+K_3)<4K_1$.

Thus, we have shown analytically that in the absence of the additive noise ξ_1 and the term $3\gamma\chi^2 y$, in a system described by Eq. (24), a noise-induced phase transition indeed occurs. As shown below, numerical simulations demonstrate that this transition remains well defined if the term $3\gamma\chi^2 y$ is included; though the additive noise ξ_1 makes it hidden. The main results of our numerical simulations are as follows.

(1) The results of numerical simulation of the complete equations (23) and (24) in the case of sufficiently broadband noise, which can be considered as white noise, are shown in Fig. 3. For comparison, the results of numerical simulation of Eq. (24) after dropping only the additive noise $\xi_1(t)$ are also given there. We call Eq. (24) with $\xi_1(t)\equiv 0$ the ''reduced equation'' and denote its solution by y_r. The solution of Eq. (25) is denoted by y_{rr}. We see that for the complete equations, which are equivalent to the initial equation, the phase transition is practically undetectable and very noisy (curve 1). For the reduced equation, the phase transition is clearly defined (curve 2). Close to the critical point the dependence of $\sigma_{y_r}^2$, which can be treated as an order parameter, on the parameter b^2, which can be regarded as temperature, is well approximated by the straight line described by the equation $\sigma_{y_r}^2=0.056(b^2-b_{cr}^2)$, where $b_{cr}\approx 4.1$. This means that the critical index is equal to 1 [see Fig. 3(a)].

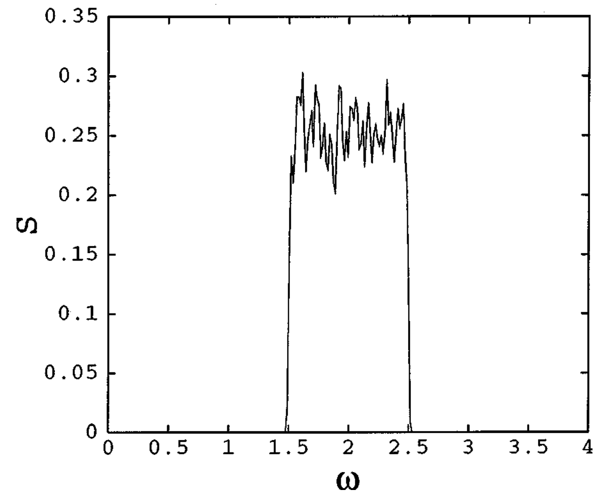

FIG. 4. The spectral density of the noise used in numerical simulations for the oscillator with quadratic nonlinearity to exclude the resonant frequency. The noise is passed through a bandpass filter.

(2) Figure 3(b) demonstrates that we can use as an order parameter not only the variance, but the mean value as well. Close to the critical point the dependence of $\langle y_r\rangle$ on b^2 can be approximated by the straight line $\langle y_r\rangle=-0.025(b^2-b_{cr}^2)$.

(3) To reveal the influence of the term $3\gamma\chi^2 y$ that was dropped in the analytical consideration, we also numerically simulated Eq. (25). The results are given in Fig. 3(a) (curve 4) and Fig. 3(b) (curve 3). We see that the phase transition occurs for a smaller value of b^2 if in the reduced equation the term $3\gamma\chi^2 y$ is ignored, i.e., this term suppresses the phase transition [compare curves 2 and 4 in Fig. 3(a)]. This is also attested by the fact that the slopes of the dependencies of σ_r^2 and σ_{rr}^2 on b^2 are essentially different. Thus, the numerical simulations have shown that in the absence of additive noise ξ_1 only, we obtain a clearly defined phase transition. As mentioned above, the additive noise ξ_1 makes the transition hidden (see the dependence for σ_y^2). It is interesting that the dependence for σ_x^2 is close to that for σ_y^2; the difference appears only for large values of the parameter b. This means that close to the critical point the influence of the noise $\chi(t)$ is negligibly small.

(4) To reduce the noise spectral density at the frequency ω_0, we have passed the noise $\xi(t)$ through a bandpass filter with central frequency $2\omega_0$ and bandwidth ω_0. The spectral density of this noise is shown in Fig. 4. We see that it is indeed very narrowband in the vicinity of ω_0. Next, we simulate Eqs. (23) and (24) using this filtered noise as $\xi(t)$. For comparison we simultaneously simulate the reduced equation (24). Figure 5 illustrates that, even though the spectral density of the filtered noise $\xi(t)$ at ω_0 is very small, the influence of the noise $\xi_1(t)$ and of the term $3\gamma\chi^2 y$ is essential. The reason is that the component of the noise $\chi(t)$ at ω_0 is not small because it is resonant. The smooth increase of

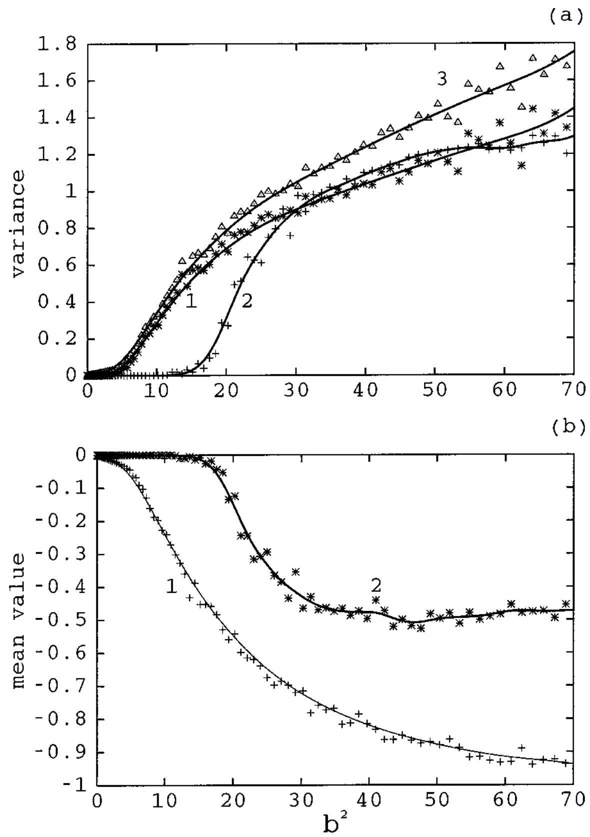

FIG. 5. Dependencies of the first moments of the simulated solutions on b^2 obtained using bandpass filtered noise. (a) Variances σ_y^2 (curve 1), $\sigma_{y_r}^2$ (curve 2), and σ_x^2 (curve 3); (b) mean values $\langle y \rangle$ (curve 1) and $\langle y_r \rangle$ (curve 2).

$\sigma_{y_r}^2$ with increasing b^2 from b_{cr}^2 onward is explained by the fact that the influence of the term $3\gamma\chi^2 y$ is less than for broadband noise.

Coming back to the initial equation (22), we decomposed the initial variable x into the sum of variables y and noise χ, which has practically no influence, since the dependencies for x and y are very similar (Figs. 3 and 5). Dropping the additive constituent of the noise from the equation for y, we get a clearly defined transition with an increase of noise intensity. From this we conclude that the initial equation describes a system in which a hidden nonequilibrium phase transition is induced by additive noise.

The transition under consideration is similar to the transition studied in the previous section, not only in the physical mechanism (autoparametrical and parametrical excitation, respectively), but also in the sense that both these transitions occur via on-off intermittency [27,28]. This is clearly visible from the shape of $y_r(t)$ [Fig. 6(b)]. Because of additive noise the intermittency for $x(t)$ is hidden [Fig. 6(a)]. As for a pendulum with randomly vibrated suspension axis and additive noise [28], the intermittency is defined more clearly for $b < b_{cr}$ [Fig. 6(a)].

At the current stage of investigation we have shown that an oscillator with quadratic nonlinearity may contain a mechanism for a phase transition induced only by additive noise. The strong influence of additive noise makes this transition undetectable in the initial equation, but we guess that it is possible to find a situation when the transition becomes well defined just by dropping some terms from the initial system equation. We leave this as an open question in the present paper.

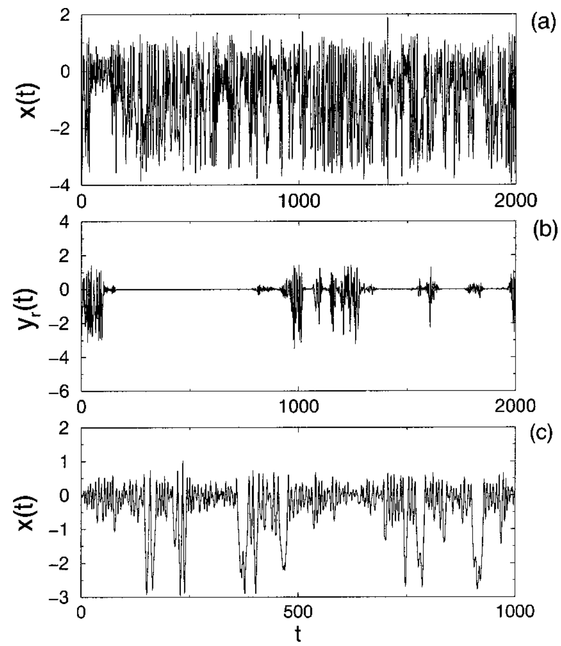

FIG. 6. A phase transition via on-off intermittency. The time series of $x(t)$ (a) and (c), and $y_r(t)$ (b) for $b^2 = 20$ (a) and (b) and $b^2 = 4$ (c). The remaining parameters are the same as in Fig. 4.

B. Electromechanical vibrator

An electromechanical vibrator energized from a source of periodic alternating current has been considered in [21,14]. It consists of a sprung plate attracted to an electromagnet with a power supply circuit forming an oscillatory circuit. We demonstrated that under certain conditions powerful low-frequency oscillations of the plate can be excited [21,14]. Below we show that similar oscillations can also be excited in the case when the power source is random. The scheme of the vibrator with a random power source is presented in Fig. 7.

The equations of this vibrator can be written as

$$\frac{d^2}{dt^2}\left(\frac{L(x)I}{L_0}\right) + 2\delta_1 \frac{dI}{dt} + \Omega_0^2 I = \xi(t),$$

$$\frac{d^2x}{dt^2} + 2\delta_2 \frac{dx}{dt} + \nu_0^2 x = F(x,I), \qquad (30)$$

where x is the plate displacement, I is the current in the oscillatory circuit, $L(x) = L_0(1 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots)$ is the inductance of the coil with a core depending on the size of the clearance between the plate and the core, $\delta_1 = R/2L_0$ and $\delta_2 = \alpha/2m$ are the damping factors for the oscillatory circuit and the plate, respectively, $\Omega_0 = 1/\sqrt{L_0 C_0}$ and $\nu_0 = \sqrt{k/m}$ are the corresponding natural frequencies, $F(x,I) = (I^2/2)(dL/dx)$ is the pondermotive force acting on the plate, and $\xi(t)$ is a random process that is proportional the electromotive force of the power source. We set $\xi(t)$ to be described by the following equation:

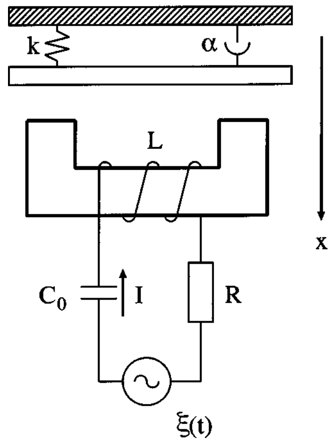

FIG. 7. A schematic image of an electromechanical vibrator with random power source. x is the plate displacement, α the friction, L the inductance, I the current, R the resistance, $\xi(t)$ the random process responsible for the electromotive force of the power source, and k the rigidity of the springs.

$$\ddot{\xi}+0.5\omega\dot{\xi}+1.125\omega^2\xi=k\chi(t), \tag{31}$$

where $\chi(t)$ is white noise. It follows from Eq. (31), that the spectral density of $\xi(t)$ peaks at the frequency ω.

Numerical simulation of Eqs. (30) shows that from a certain value of the power source intensity, low-frequency oscillations of the plate appear. The dependence of the variance of these oscillations (σ_x^2) on k^2, which is proportional to the noise intensity, is illustrated in Fig. 8(a). The form of this dependence closely resembles the corresponding dependence for a pendulum with a randomly vibrated suspension axis and additive noise (see Fig. 1). We find from this plot that, for sufficiently large values of k^2, the dependence can be approximated by a straight line described by the equation $\sigma_x^2=0.3(k^2-0.025)$. Taking into account the similarity with the dependencies for noise-induced transitions in a pendulum, we can take the point where this straight line crosses the abscissa as the threshold of a noise-induced transition. Hence, the critical value of k is equal to 0.158. Unlike the variance σ_x^2, the variance of the current fluctuations ($\sigma_I^2=\overline{I^2}$) increases with an approximately constant rate as k^2 increases. The corresponding dependence is presented in Fig. 8(b) (curve 1). It can be approximated by the straight line $\sigma_I^2=0.075k^2$. Owing to the presence of a quadratic nonlinearity, the mean value of the plate displacement is nonzero. The dependence of $\overline{x}$ on k^2 is also shown in Fig. 8(b) (curve 2).

Typically for noise-induced transitions that lead to the excitation of oscillations [28], for $k<k_{\mathrm{cr}}$ one can detect on-off intermittencylike behavior in oscillations of the variable x [see, for example, Fig. 9(a)]. With increase of k this effect disappears. An example of the oscillations of x, I, and ξ for $k>k_{\mathrm{cr}}$ is given in Fig. 9(b).

Power spectra of the random source and excited oscillations are shown in Fig. 10. It is clearly seen that we deal with high-frequency excitation. The mechanism responsible for the excitation seems to be similar to combination resonance.

As in the case of a pendulum with slight additive noise, noise-induced oscillations of the vibrator under consideration can be partially suppressed by additional harmonic action [27]. But, in contrast to the pendulum, the suppression occurs at low-frequency action rather than at high frequency. If the action frequency is high, the action has little or no effect on the variance of the plate oscillations. To describe the additional action, we add the term $a\cos\omega t$ to $\xi(t)$ on the right of the first equation of Eq. (30). Under low-frequency action a considerable constant displacement of the plate appears. Therefore, the study of the suppression is conveniently performed using the variance of the plate velocity instead of the plate displacement. The dependencies of this variance (σ_y^2) on the action amplitude a for a fixed value of the action frequency ω and on ω for a fixed value of a are shown in Fig. 11. We see that for a fixed value of the frequency ($\omega=0.2$) the variance σ_y^2 initially decreases as the action amplitude increases, and then abruptly increases owing to excitation of oscillations at the frequency ω. For a fixed value of the action amplitude, the dependence of σ_y^2 on ω has a minimum whose location depends on the amplitude a [Fig. 11(b)].

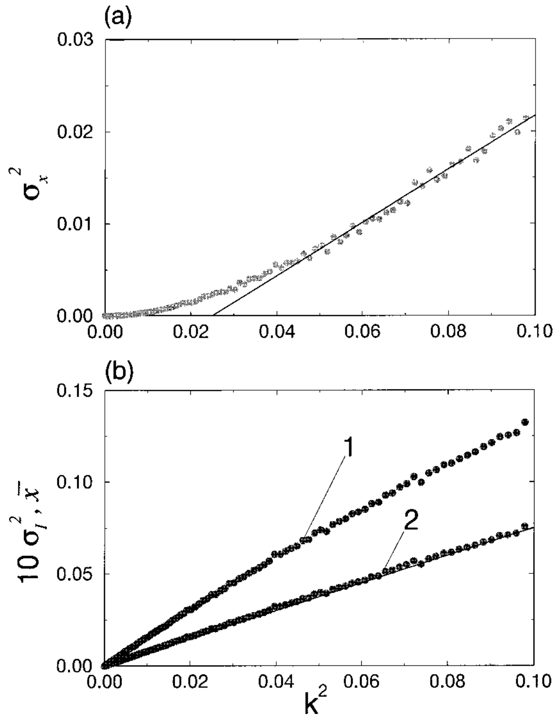

FIG. 8. The noise-induced transition in the electromechanical vibrator caused by a mechanism similar to combination resonance. The dependencies of σ_x^2 (a) and $10\sigma_I^2$ and $\overline{x}$ (b) on k^2 for $\omega=1$, $\Omega_0=0.9$, $\nu_0=0.05$, $\delta_1=0.1$, $\delta_2=0.005$, $L_0/m=0.1$, $a_1=1$, $a_2=-0.5$, and $a_3=0.1$.

IV. TRANSITIONS INDUCED BY BOTH MULTIPLICATIVE AND ADDITIVE NOISE: STABILIZATION OF NOISE-INDUCED OSCILLATIONS BY ADDITIVE NOISE

In this section we consider an example of a system under the combined action of additive and multiplicative noise. Both multiplicative and additive noise can induce a transition, and, what is especially interesting, a combination of their actions stabilizes noise-induced oscillations. To demon-

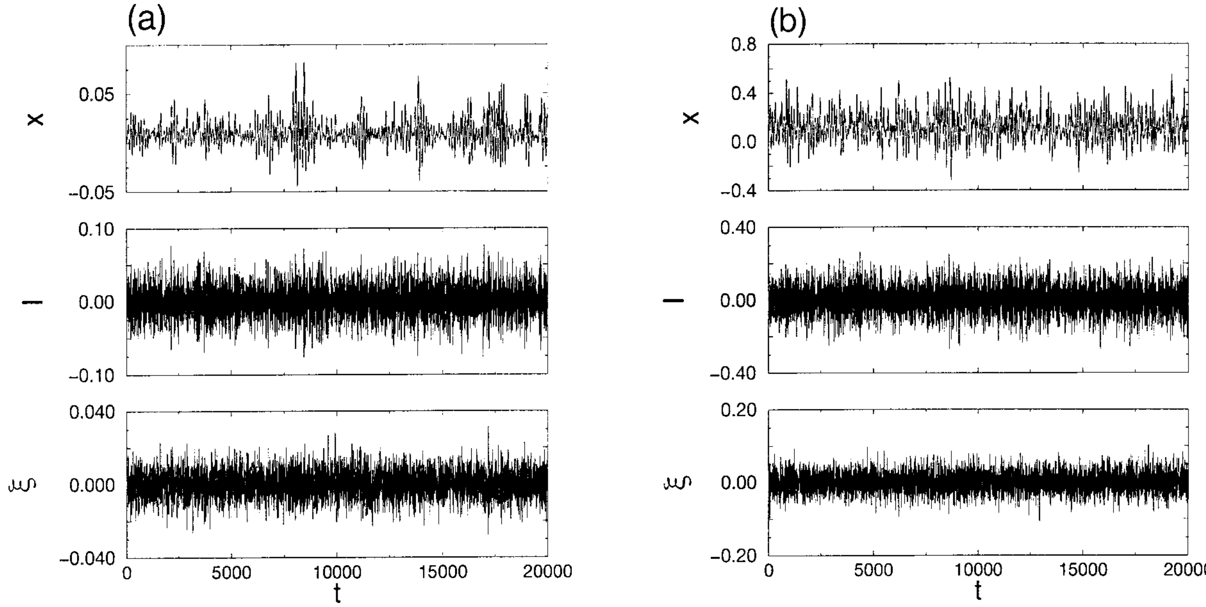

FIG. 9. On-off intermittency in the vibrator. Examples of oscillations of the plate (x), of the current in the oscillatory circuit (I), and of the power source (ξ) for $k=0.08$ (a) and $k=0.3$ (b).

strate these effects, we use a standard epidemiological model for the description of seasonal oscillations of childhood infections, such as chickenpox, measles, mumps, and rubella, under the influence of variations of the contact rate of children susceptible to infection with infective children. This model has been studied in detail both in the case of periodic variation of the contact rate [22–24,29] and in the case of random variation of the contact rate [15,29]. Here we dwell only on one important aspect of this problem, namely, on the stabilizing influence of a combination of additive and multiplicative noise on the excitation of induced oscillations.

The model equations are

$$\dot{S}=m(1-S)-bSI, \quad \dot{E}=bSI-(m+a)E,$$

$$\dot{I}=aE-(m+g)I, \tag{32}$$

$$\dot{R}=gI-mR, \tag{33}$$

where S is the relative number of children susceptible to infection, E is the relative number of children exposed but not yet infective, I is the number of infective children, R is the number of children recovered and immune, $1/m$ is the average expectancy time, $1/a$ is the average latency period, $1/g$ is the average infection period, and b is the contact rate (the average number of susceptibles in contact yearly with infectives). Let us note that Eqs. (32) do not contain the variable R; hence these equations can be considered independently of Eq. (33).

It is easy to show that Eqs. (32) for $b=b_0=\text{const}$, and for any values of the remaining parameters, have one aperiodically unstable singular point with coordinates $S=1,E=I=0$, and one stable singular point with coordinates

$$S_0=\frac{(m+a)(m+g)}{ab_0}, \quad E_0=\frac{m}{m+a}-\frac{m(m+g)}{ab_0},$$

$$I_0=\frac{am}{(m+a)(m+g)}-\frac{m}{b_0}. \tag{34}$$

If the parameter b varies with time then the variables S, E, and I will oscillate, and these oscillations will be executed around the stable singular point with coordinates (34). Therefore, it is convenient to substitute into Eqs. (32) the new variables $x=S/S_0-1$, $y=E/E_0-1$, and $z=I/I_0-1$. Putting $b=b_0[1+b_1 f(t)]$, where $f(t)$ is a function describing the shape of the contact rate variation, we rewrite Eqs. (32) in the variables x, y, z:

$$\dot{x}+mx=-b_0 I_0[1+b_1 f(t)](x+z+xz)-b_0 b_1 I_0 f(t),$$

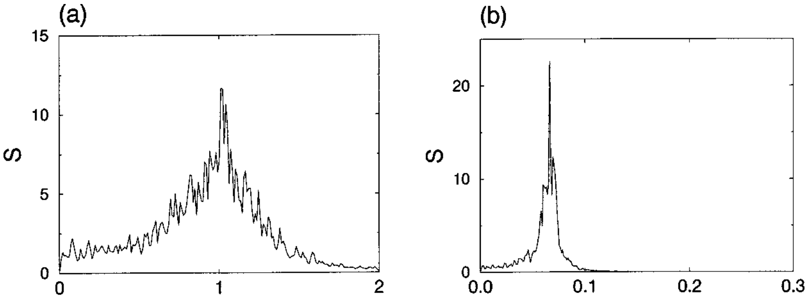

FIG. 10. The power spectra of the random power source (a) and of the solution x (b).

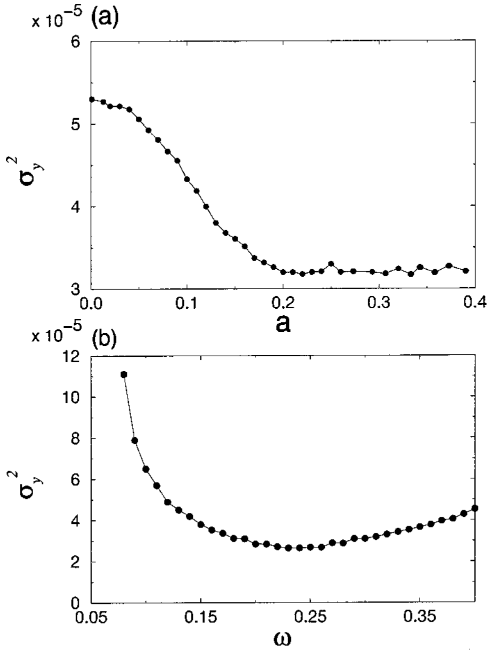

FIG. 11. The dependencies of the variance σ_y^2 on the action amplitude a for $\omega=0.2$ (a) and on the action frequency ω for $a=0.3$ (b). For $\omega=0.2$, $a>0.4$, the variance abruptly increases owing to excitation of oscillations at the frequency ω and goes, in fact, to infinity.

$$\dot{y}+(m+a)y=(m+a)[1+b_1 f(t)](x+z+xz)+(m+a)b_1 f(t), \tag{35}$$

$$\dot{z}+(m+g)z=(m+g)y.$$

In Eqs. (35) the term $b_1 f(t)$ can be considered as an external action upon the system. This form of the equations clearly shows that this action is not only multiplicative, i.e., parametric, but additive as well.

Olsen and Schaffer [23] set the following values of the parameters: $m=0.02\,\text{year}^{-1}$, $a=35.84\,\text{year}^{-1}$, $g=100\,\text{year}^{-1}$, $b_0=1800\,\text{year}^{-1}$, and $b_1=0.28$. These parameters correspond to estimates made for childhood diseases in first world countries. We follow them.

In [15] we supposed that the contact rate b varies randomly with the main period equal to one year, i.e., $f(t)=\chi(t)$, where $\chi(t)$ is a random process that is a solution of the equation

$$\ddot{\chi}+2\pi\dot{\chi}+6\pi^2\chi=k\xi(t), \tag{36}$$

$\xi(t)$ is white noise, and k is a factor that we choose so that the variance of $\chi(t)$ is equal to 1/2. It is easily seen that the spectral density of $\chi(t)$ peaks at the frequency $\omega=2\pi$.

Noise-induced oscillations appear as a result of a noise-induced phase transition. To show this let us consider Fig. 12, where the dependence of σ_x^2 on b_1 is presented. With an increase of noise intensity, the intensity of noise-induced oscillations is increased too. For rather large $b_1>b_{cr}$ this dependence can be approximated by a straight line. The intersection point of this line and the abscissa can be taken as a point of a transition—a threshold value b_{cr}. To prove it let us drop the artificially multiplicative noise from Eqs. (35). In

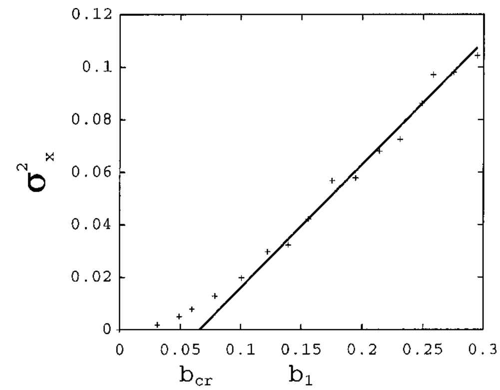

FIG. 12. A noise-induced phase transition in the SEIR model. The dependence of σ^2 on the parameter b_1 in the case of a random variation of the contact rate. The solid line represents $\sigma^2=0.47(b_1-0.066)$.

this case the variance of oscillations is equal to zero for $b_1<b_{cr}$ and goes to infinity shortly after the noise intensity represented by the parameter b_1 exceeds its critical value. The same situation is observed if additive noise is absent but multiplicative noise is present. Now it is clear that the point $b_1=b_{cr}$ is a point of noise-induced phase transition, which can be induced by both multiplicative and additive noise. The physical mechanisms responsible for this effect are likely to be the same as for the pendulum (Sec. II) and the nonlinear oscillator (Sec. III A), respectively.

It is even more interesting that the combined action of additive and multiplicative noise performs a stabilization of noise-induced oscillations: in this case the dependence of variance on noise intensity does not go to infinity. Again, as for previously considered models [28], the transition can be accompanied by the effect of on-off intermittency. In the absence of additive noise one can observe on-off intermittency near the threshold (Fig. 13).

V. CONCLUSIONS

We have studied in this paper the role of additive noise in noise-induced phase transitions and have shown that it can be nontrivial. We have found several phenomena by consideration of different typical models; each of them has demonstrated a certain aspect of the problem. Consideration of a pendulum under the action of multiplicative and additive noise has shown that, if a noise-induced transition occurs in

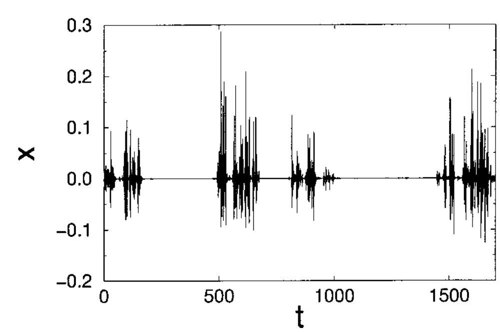

FIG. 13. On-off intermittency in the SEIR model. An example of oscillations of the variables x and y for $b_1=0.099$ for the case of multiplicative random action alone.

the presence of additive noise, it is blurred by this noise and becomes hidden. We have presented results of an analytical study confirmed by numerical simulations. By the examples of Sec. III we have demonstrated that there are mechanisms which allow additive noise alone to induce a hidden transition. Consideration of an epidemiological model has shown that, moreover, there exist nonlinear systems in which only the combined action of multiplicative and additive noise causes stable noise-induced oscillations. In such systems the joint influence of additive and multiplicative noise can be interpreted as the stabilization of noise-induced oscillations. In the present study we have considered only transitions that lead to the excitation of oscillations (e.g., in contrast to [20,30,31]). It should be mentioned also that we have recently shown in [19,20] that the role of additive noise may also be crucial in noise-induced transitions that lead to the creation of a mean field in a spatially extended system.

ACKNOWLEDGMENTS

A.Z. acknowledges financial support from MPG and P.L. acknowledges support from DFG (SFB 555).

APPENDIX A: LANGEVIN EQUATIONS

The following Langevin equations can be related to the Fokker-Planck equation (5) in view of Eqs. (10) and (11):

$$\dot{A}=\beta\left(\eta-\frac{3\omega_0^2}{4}\alpha A^2\right)A+\frac{\omega_0^2}{2A}K_{12}+\frac{\omega_0}{2}A\zeta_{11}(t)+\omega_0\zeta_{12}(t),$$

$$\dot{\phi}=\omega_0^2 M+\omega_0\left(\zeta_{21}(t)+\frac{\zeta_{22}(t)}{A}\right), \tag{A1}$$

where $\zeta_{11}(t)$, $\zeta_{12}(t)$, $\zeta_{21}(t)$, and $\zeta_{22}(t)$ are white noises with zero mean value and uncorrelated with A. The intensities of these noises are K_{11}, K_{12}, K_{21}, and K_{22}, respectively. We note that even in the case with $\kappa_{\xi_2}=0$ Eqs. (37) differ from that derived in [26]. The reason is that there the variable $u=\ln A$ in place of A was used, i.e., the correlation between the noise $\xi(t)$ and the amplitude A was implicitly ignored [14,1,25].

APPENDIX B: CALCULATIONS IN THE CASE WITH ADDITIVE NOISE

The dependence of the mean amplitude squared on the multiplicative noise intensity in the case where additive noise also acts on the pendulum can be calculated as follows. Upon integrating Eq. (19) over ϕ and calculating the integral within the exponential, we obtain

$$w(A)=2\pi A w(A,\phi)=CA^2(A^2+q/a)^{3(q-1)/2(1+\eta)}\times\exp\left(-\frac{3aA^2}{2(1+\eta)}\right). \tag{B1}$$

It follows from the normalization condition that

$$C^{-1}=\int_0^\infty A^2(A^2+q/a)^{3(q-1)/2(1+\eta)}\exp\left(-\frac{3aA^2}{2(1+\eta)}\right)dA. \tag{B2}$$

The integral on the right-hand side of Eq. (B2) can be expressed in terms of a Whittaker function [32]. As a result we find

$$C^{-1}=\frac{\sqrt{\pi}}{4a^{2\mu}q^{1/2-\mu}}\left(\frac{3}{2(1+\eta)}\right)^{-\mu-1/2}\exp\left(\frac{3q}{4(1+\eta)}\right)\times W_{\mu-1,\mu}\left(\frac{3q}{2(1+\eta)}\right), \tag{B3}$$

where $\mu=3(\eta+q)/4(1+\eta)$.

We obtain the expression for C in explicit form in the limiting case when the additive noise intensity is small compared to that of the multiplicative noise, so that

$$q\ll 1. \tag{B4}$$

In this case we can use a representation of the Whittaker function $W_{\lambda,\mu}(z)$ in terms of two other Whittaker functions $M_{\lambda,\mu}(z)$ and $M_{\lambda,-\mu}(z)$ [32]:

$$W_{\lambda,\mu}(z)=\frac{\Gamma(-2\mu)}{\Gamma(1/2-\mu-\lambda)}M_{\lambda,\mu}(z)+\frac{\Gamma(2\mu)}{\Gamma(1/2+\mu-\lambda)}M_{\lambda,-\mu}(z). \tag{B5}$$

We then expand each of the functions $M_{\lambda,\mu}(z)$ and $M_{\lambda,-\mu}(z)$ in powers of z [32]:

$$W_{\lambda,\mu}(z)=\sqrt{z}\exp\left(-\frac{z}{2}\right)\left[\frac{\Gamma(-2\mu)}{\Gamma(1/2-\mu-\lambda)}z^{\mu}\times\left(1+\frac{1-2(\lambda-\mu)}{2(1+2\mu)}z+\cdots\right)+\frac{\Gamma(2\mu)}{\Gamma(1/2+\mu-\lambda)}z^{-\mu}\left(1+\frac{1-2(\lambda+\mu)}{2(1-2\mu)}z+\cdots\right)\right]. \tag{B6}$$

Substituting Eq. (B6) into Eq. (B3) we get

$$C^{-1}=\frac{\sqrt{\pi}}{4a^{2\mu}}\left[\frac{\Gamma(-2\mu)}{\Gamma(3/2-2\mu)}q^{2\mu}\left(1+\frac{9q}{4(1+2\mu)(1+\eta)}+\cdots\right)+\frac{\Gamma(2\mu)}{\Gamma(3/2)}\left(\frac{2(1+\eta)}{3}\right)^{2\mu}\times\left(1+\frac{3(3-4\mu)q}{4(1-2\mu)(1+\eta)}+\cdots\right)\right]. \tag{B7}$$

The expression (14), obtained in the absence of additive noise, follows at once from Eq. (B7) for $q\to 0$.

The probability distribution (B1) for $q\neq 0$ differs essentially from Eq. (15): first, it is not a δ function for $\eta<0$ and, secondly, $w(A)=0$ for $A=0$.

Using Eqs. (B1) and (B3) we can calculate $\langle A\rangle$ and $\langle A^2\rangle$. For example, for $\langle A^2\rangle$ we obtain

$$a\langle A^2\rangle=\sqrt{\frac{3q(1+\eta)}{2}}\,\frac{W_{\mu-3/2,\mu+1/2}[3q/2(1+\eta)]}{W_{\mu-1,\mu}[3q/2(1+\eta)]}. \tag{B8}$$

Taking into account the recursion relation [32]

$$W_{\lambda,\mu}(z)=\sqrt{z}W_{\lambda-1/2,\mu+1/2}(z)+(\tfrac{1}{2}-\lambda-\mu)W_{\lambda-1,\mu}(z),$$

the expression (B8) can be rewritten as

$$a\langle A^2\rangle=(1+\eta)\left(1-(\tfrac{3}{2}-2\mu)\frac{W_{\mu-2,\mu}[3q/2(1+\eta)]}{W_{\mu-1,\mu}[3q/2(1+\eta)]}\right). \tag{B9}$$

The expression for $\langle A^2\rangle$ can be obtained in explicit form only with the constraint (B4). Using Eq. (B6) we find for $W_{\mu-2,\mu}(z)/W_{\mu-1,\mu}(z)$ the following approximate expression:

$$\frac{W_{\mu-2,\mu}(z)}{W_{\mu-1,\mu}(z)}\approx\frac{2}{(3-4\mu)}\Bigg[\frac{\sqrt{\pi}}{2}\Gamma(-2\mu)z^{\mu}(1-2\mu)\times[2(1+2\mu)+5z]+\Gamma(2\mu)\Gamma(\tfrac{3}{2}-2\mu)\times\left(1-\frac{4\mu}{3}\right)z^{-\mu}(1+2\mu)[2(1-2\mu)+(5-4\mu)z]\Bigg]\Bigg(\frac{\sqrt{\pi}}{2}\Gamma(-2\mu)z^{\mu}(1-2\mu)[2(1+2\mu)+3z]+\Gamma(2\mu)\Gamma(\tfrac{3}{2}-2\mu)z^{-\mu}\times(1+2\mu)[2(1-2\mu)+(3-4\mu)z]\Bigg)^{-1}. \tag{B10}$$

Substituting Eq. (B10) in Eq. (B9) we get the required Eq. (20).

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[16] We call the transitions considered in this paper noise-induced *phase* transitions. In our terminology we follow the concept suggested by Haken in [33], where he shows that the name *phase transitions* can be used for the transitions considered by us on the basis of analogy with phase transitions in equilibrium systems. Such a name expresses that these new transitions are closely akin to the classical equilibrium phase transitions and to a more recent class of nonequilibrium phase transitions (see also the Introduction in [9]). However, there is also another point of view [13,12], arguing that the name noise-induced *phase* transitions can be used only for transitions in spatially extended systems.
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