Spatial patterns induced by additive noise

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(Received 18 May 1998)

We consider a nonlinear lattice with spatial coupling under the influence of multiplicative and additive noise. In contrast to other studies, we pay attention mainly to the role of the additive noise and show that additive noise, much like multiplicative noise, is able to induce spatial patterns. The reason is that the increase of additive noise causes a nonequilibrium phase transition that manifests itself in the formation of ordered spatial patterns. The presence of the additive noise correlated or uncorrelated with the multiplicative noise is a necessary condition of the phase transition. We review the mean field theory for this model and show that this theory predicts a reentrant phase transition caused by additive noise. Theoretical predictions are confirmed by numerical simulations. [S1063-651X(98)12510-2]

PACS number(s): 05.40.+j, 47.54.+r, 05.70.Fh

I. INTRODUCTION

Over the past two decades nonlinear systems with noise have been continuously attracting attention. The reason is the ordering role of noise in such phenomena as stochastic resonance [1], noise-induced transport [2], or noise-induced transitions [3]. A large variety of models [4–12] appear to demonstrate nonequilibrium noise-induced phase transition. In these studies only multiplicative noise is shown to be the reason for the transition and much less attention has been paid to the role of additive noise. Recently, we started to study the influence of an additive noise on noise-induced transitions. It was shown that this influence can be crucial because the additive noise may shift the boundaries of the noise-induced phase transition [13] or even cause these transitions [13,14].

In the present paper we continue to study the influence of additive noise on noise-induced phase transitions. We consider the role of the additive noise in the formation of the ordered spatially inhomogeneous patterns. For this we investigate a paradigmatic model introduced in [7] (for the history of the subject see also [15–18]). As noted in [19], investigation of this model is helpful for the understanding of results of experiments on electrohydrodynamic convection in nematic liquid crystals with thermal fluctuations (additive noise) and an external stochastic voltage (multiplicative noise). We show that this model displays noise-induced spatial patterns with an increase of additive noise. After exceeding an optimal level of the additive noise a further increase destroys the structures again.

First we review mean field theory for this model [7]. The theory predicts the existence of the reentrant phase transition by increasing the additive noise for two limiting cases of correlation between both additive and multiplicative noises. The transition manifests itself in breaking the symmetry and appearing ordered spatial structures. Next we perform numerical calculations and confirm some results of the theoretical considerations. After a discussion about understanding of the phenomena observed we summarize results obtained.

II. MODEL AND MEAN FIELD THEORY

We consider a scalar field $x_{\mathbf{r}}$ defined on a spatial lattice with points $\mathbf{r}$. The time evolution of the field is described by a set of Langevin equations [7]

$$\dot{x}_{\mathbf{r}} = f(x_{\mathbf{r}}) + g(x_{\mathbf{r}})\xi_{\mathbf{r}} + \mathcal{L}x_{\mathbf{r}} + \zeta_{\mathbf{r}}, \tag{1}$$

with f and g defined as

$$f(x) = -x(1+x^2)^2, \quad g(x) = a^2 + x^2 \tag{2}$$

and $\xi_{\mathbf{r}}, \zeta_{\mathbf{r}}$ independent zero-mean-value Gaussian white noise sources

$$\langle \xi_{\mathbf{r}}(t)\xi_{\mathbf{r}'}(t')\rangle = \sigma_\xi^2 \delta_{\mathbf{r},\mathbf{r}'}\delta(t-t'), \tag{3}$$

$$\langle \zeta_{\mathbf{r}}(t)\zeta_{\mathbf{r}'}(t')\rangle = \sigma_\zeta^2 \delta_{\mathbf{r},\mathbf{r}'}\delta(t-t'). \tag{4}$$

We note that such a form of the function $g(x)$ implies that the parameter a is responsible for an additive noise strongly correlated with the multiplicative one. To gain knowledge about the influence of additive noise on the noise-induced phase transition we study two different problems. First the constant contribution a^2 of the multiplicative noise $\xi_{\mathbf{r}}$ is changed, setting $\sigma_\zeta^2 = 0$. The origin one could see, for instance, in a decomposition of the multiplicative noise into two parts $g(x)\xi_{\mathbf{r}} = a^2\xi_{\mathbf{r}}^1 + x^2\xi_{\mathbf{r}}^2$. Changing the parameter a would imply an increase or a decrease of additive noise $a^2\xi_{\mathbf{r}}^1$ strongly correlated with the multiplicative one. We prove that the constant contribution of that noise is essential for the nonequilibrium phase transition. Only in the presence of the additive component with an optimally selected value does the system exhibit spatially disordered states.

A different situation is the variation of the noise intensity σ_ζ^2. It models additive noise independently of the multiplicative one. In that case we set $a=0$. Again we will find a strong influence of the additive noise ζ.

The spatial coupling in the model is described by the coupling operator $\mathcal{L}$ [see Eq. (1)], which is a discretized version of the Swift-Hohenberg coupling term $-D(q_0^2+\nabla^2)^2$:

$$\mathcal{L}x_{\mathbf{r}} = -D\left\{q_0^2 - \frac{1}{\Delta^2}\sum_{i=1}^{2d}\left[1-\exp\left(\Delta \mathbf{e}_i \cdot \frac{\partial}{\partial \mathbf{r}}\right)\right]\right\}^2 x_{\mathbf{r}}. \tag{5}$$

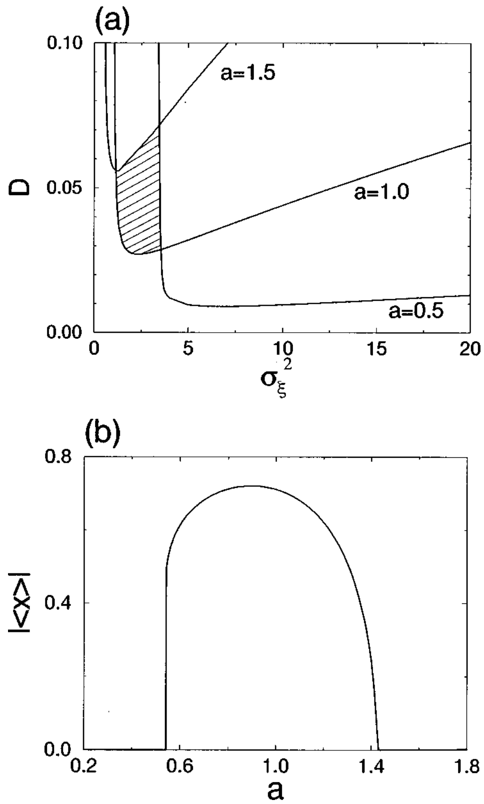

FIG. 1. (a) Boundaries of the phase transition on the plane (σ_ξ^2, D) in the case of correlated additive noise. The values of parameter a are shown in the picture. (b) Dependence of the order parameter $|\langle x\rangle|$ on the control parameter a for $D=0.06$ and $\sigma_\xi^2 = 3.0$.

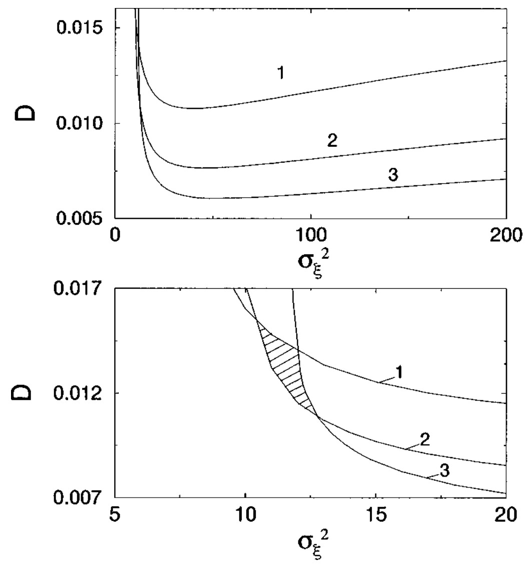

FIG. 2. Boundaries of the phase transition on the plane (σ_ξ^2, D) in the case of uncorrelated additive noise. The parameter σ_ζ^2 is equal to 1.0 (label 1), 0.5 (label 2), and 0.3 (label 3).

Here $\mathbf{e}_i$ are the unit vectors associated with the cubic lattice of the dimension d, and Δ is the lattice space.

The conditions of phase transition can be found using generalized Weiss mean field theory [7]. According to this theory, we replace the value of the scalar variable $x_{\mathbf{r}'}$ at the sites coupled to $x_{\mathbf{r}}$ by its averaged value, assuming the specific nonuniform average field

$$\langle x_{\mathbf{r}'}\rangle = \langle x\rangle \cos[\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')]. \tag{6}$$

Substituting Eq. (6) into Eq. (5) we get for $x_{\mathbf{r}}$

$$\dot{x} = f(x) + g(x)\xi + D\omega(\mathbf{k})x - D_{\mathrm{eff}}(x-\langle x\rangle) + \zeta, \tag{7}$$

where

$$D_{\mathrm{eff}} = \left[\left(\frac{2d}{\Delta^2} - q_0^2\right)^2 + \frac{2d}{\Delta^2} + \omega(\mathbf{k})\right] D \tag{8}$$

and

$$\omega(\mathbf{k}) = -D\left[q_0^2 - \frac{2}{\Delta^2}(2-\cos k_x\Delta - \cos k_y\Delta)\right]^2. \tag{9}$$

The expression for $\omega(\mathbf{k})$ can be obtained if one considers how $\mathcal{L}$ acts on a plane wave $e^{i\mathbf{k}\cdot\mathbf{r}}$ for the case of a two-dimensional lattice:

$$\mathcal{L}e^{i\mathbf{k}\cdot\mathbf{r}} = \omega(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}}. \tag{10}$$

Note that for $|\mathbf{k}| \ll 2\pi/\Delta$ the dispersion relation $\omega(\mathbf{k})$ reduces to the relation for the continuous Swift-Hohenberg model: $-D(q_0^2 - |\mathbf{k}|^2)^2$. For the most unstable mode in the discrete case $\omega(\mathbf{k})=0$ (see [7]).

Now the value $\langle x\rangle$ plays the role of the amplitude of the spatial patterns with an effective diffusion coefficient D_{eff}. The Fokker-Planck equation corresponding to Eq. (7) in the case $\omega(\mathbf{k})=0$ is

$$\frac{\partial w}{\partial t} = -\frac{\partial}{\partial x}\left[[f(x) - D_{\mathrm{eff}}(x-\langle x\rangle)]w - \frac{\sigma_\xi^2}{2}\left(g(x)\frac{\partial}{\partial x}[g(x)w]\right) - \frac{\sigma_\zeta^2}{2}\frac{\partial w}{\partial x}\right].$$

For this equation it is possible to find the exact steady state probability, parametrically dependent on $\langle x\rangle$:

$$w_{\mathrm{st}}(x) = \frac{C(\langle x\rangle)}{\sqrt{\sigma_\xi^2 g^2(x) + \sigma_\zeta^2}} \exp\left(2\int_0^x \frac{f(y) - D_{\mathrm{eff}}(y-\langle x\rangle)}{\sigma_\xi^2 g^2(y) + \sigma_\zeta^2} dy\right), \tag{11}$$

where $C(\langle x\rangle)$ is the normalization constant determined by

$$C^{-1}(\langle x\rangle) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\sigma_\xi^2 g^2(x) + \sigma_\zeta^2}} \times \exp\left(2\int_0^x \frac{f(y) - D_{\mathrm{eff}}(y-\langle x\rangle)}{\sigma_\xi^2 g^2(y) + \sigma_\zeta^2} dy\right) dx. \tag{12}$$

For the value $\langle x\rangle$ we obtain

$$\langle x\rangle = \int x w_{\mathrm{st}}(x, \langle x\rangle) dx, \tag{13}$$

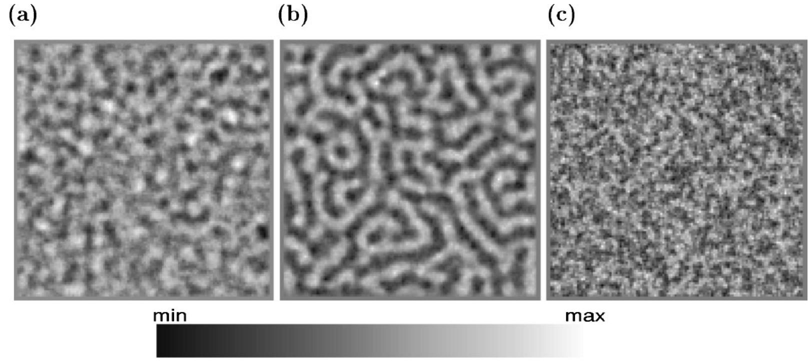

FIG. 3. Snapshots of the field for $D=1.0$, $\sigma_\xi^2=1.8$, and $\sigma_\zeta^2=0$. The parameter a is equal to (a) 0.1, (b) 1.0, and (c) 10.0. The increase of the additive noise induces spatial patterns. The scalar field from minimum to maximum value is coded in accordance with the color scale shown in the same figure.

which is nonlinear equation for the unknown value $\langle x\rangle$ and closes the system of equations.

Solving Eq. (13), we can calculate boundaries of phases with $\langle x\rangle\neq 0$ (order) and $\langle x\rangle=0$ (disorder) for specific $\mathbf{k}$ whose modes are excited first. Nonzero solution of Eq. (13) means excitation of the corresponding mode and hence existence of the phase transition. Due to the special form of the spatial coupling, the transition manifests itself in a formation of ordered spatial patterns with the wave number defined by the parameter q_0.

The computation of Eq. (13) shows that the condition for the existence of nonzero solutions is

$$\left|\frac{dF}{dm}\right|_{m=0}\geqslant 1. \tag{14}$$

We note that for rather large D four nonzero roots (two stable and two unstable) of Eq. (13) may be observed. It is an open question whether this indicates that additionally also noise-induced first-order phase transition may be found in this model (to this point see also [19,20]).

III. ADDITIVE NOISE AND NOISE-INDUCED TRANSITION

First we study the case if an additive noise is strongly correlated with multiplicative noise (in this case $\sigma_\zeta^2=0$). For different values of a the boundary of the phase transition on the plane (σ_ξ^2,D) is shown in Fig. 1. As it is seen from this plot, the reentrant phase transition occurs for the specific value of a with the increase of σ_ξ^2 [7]. Solving Eq. (13) for other values of a, we find that as a decreases the boundary of the phase transition significantly drops and is right shifted (see Fig. 1). Hence there is a set of parameters (σ_ξ^2,D) for which the reentrant phase transition occurs with the increase of a (dashed region in Fig. 1). This means that for fixed values of σ_ξ^2 and D an increase of additive noise intensity will first induce the spatial patterns and then destroy them. We note that this phase transition is possible only in the presence of multiplicative noise. The corresponding dependence of the order parameter $|\langle x\rangle|$ on the control parameter a is shown in Fig. 1(b).

Now we study the case where the additive noise is uncorrelated (independent) from the multiplicative noise ($a=0$, $\sigma_\zeta^2\neq 0$). As Fig. 2 shows in this case the behavior of the system is qualitatively the same: For fixed parameters (D,σ_ζ^2) an increase of the multiplicative noise intensity σ_ξ^2 causes the noise-induced phase transition. Hence for large enough coupling D one expects the formation of the spatially ordered patterns if σ_ξ^2 exceeds it critical value. As concerns the influence of the additive noise on the transition, an amplification of the additive noise intensity shifts the transition boundaries and therefore causes the reentrant disorder-order-disorder nonequilibrium phase transition. It can be clearly seen if one take a point with fixed parameters (D,σ_ξ^2) from the dashed region in the Fig. 2: With an increase of σ_ζ^2 this point first belongs to the disordered phase, then to the ordered one, and then again to the disordered phase.

IV. NUMERICAL SIMULATIONS

We check the relevance of the theory presented above by numerical simulations of the initial equations (1). We use an Euler scheme for stochastic differential equations interpreted in the Stratonovich sense [21,22]. The time step has been set $\Delta t=5\times 10^{-4}$. For simulations we integrate the scalar field $x_{\mathbf{r}}(t)$ on a two-dimensional square lattice 128×128 with conditions $x_{\mathbf{r}}=0$ and $\mathbf{n}\cdot\boldsymbol{\nabla}x_{\mathbf{r}}=0$ at the boundaries. Here $\mathbf{n}$ is the vector normal to the boundary.

First we set $\sigma_\zeta^2=0$ and $a\neq 0$. The remaining parameters are $D=1$, $q_0=0.7$, $\Delta=0.5$, and $\sigma_\xi^2=1.8$. For these values the mean field theory predicts the existence of spatial patterns of the most unstable mode $|k|=1.0478$ for $a=1$. For additive noise intensities significantly larger than this value, for example, $a=10.0$, or significantly smaller, $a=0.1$, according to the mean field theory no spatial patterns will be exhibited.

In Fig. 3 the picture of the field after 100 time units has been plotted for three different noise intensities. Clearly one can see the appearance of the spatial patterns with the increase of the additive noise and its further destruction. These calculations confirm the predictions of the mean field theory for the case of correlated additive noise.

The ordered patterns in Fig. 3(b) have rotational symmetry, which can be clearly observed in the two-dimensional

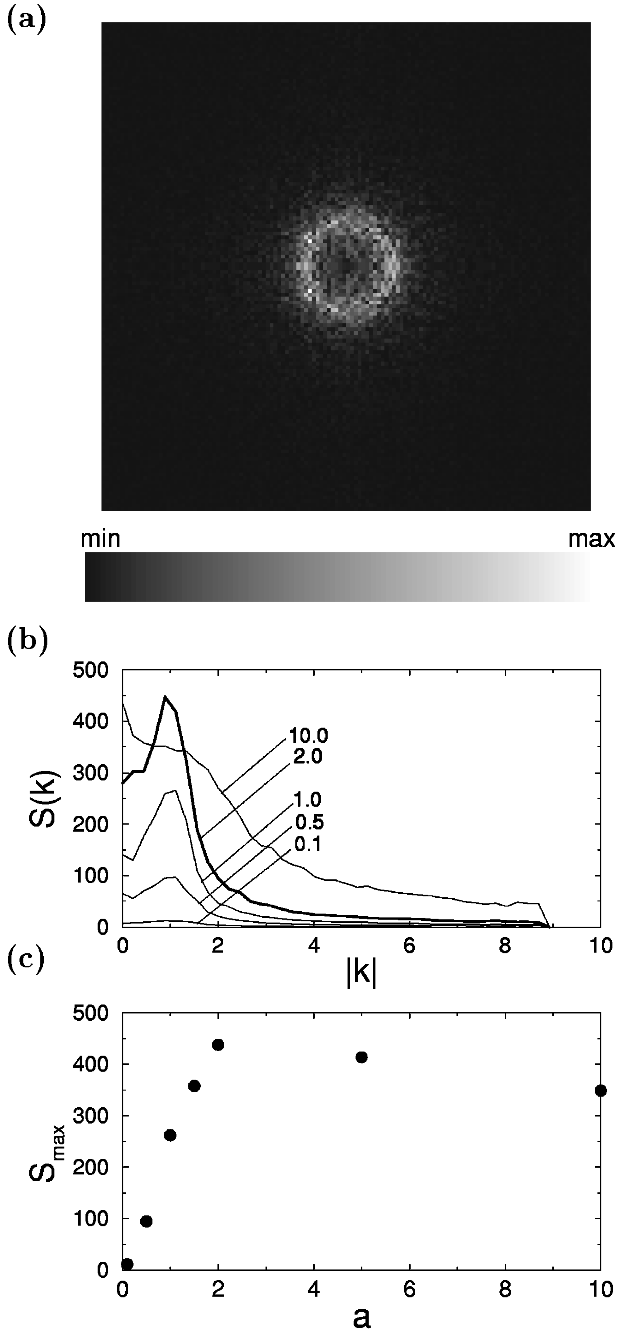

FIG. 4. (a) 2D Fourier transform of the pattern shown in Fig. 3(b). Rotational symmetry is observed. (max,min) values are (1337,0.1). (b) Fourier transform averaged over angles for $D=1.0$ and $\sigma_\xi^2=1.8$. Values of parameter a are shown in the figure. (c) Dependence of S_{max} on a.

Fourier transform of the field represented in Fig. 4. To make the transition more evident we have plotted the Fourier transform of the field averaged over the angles of the wave vector. It is shown in Fig. 4 for different values of a. With an increase of a a maximum in this structure function is found. It corresponds to the dominating value $|k|_{\text{max}}$, indicating the appearance of a spatial pattern with a wavelength $2\pi/|k|_{\text{max}}$. After an optimal value of a the maximum of the structure function disappears, again signaling the destruction of the order.

Next we consider the case of uncorrelated additive noise, in which $a=0$ and $\sigma_\zeta^2 \neq 0$. Numerical simulations show that the behavior of the model is quite similar to the case of the correlated additive noise. An increase of the additive noise causes the formation of the rotationally symmetric spatial patterns. A further increase of the additive noise destroys this pattern (see Fig. 5). These results are also in good agreement with the predictions of mean field theory.

V. DISCUSSION

Now we discuss the mechanism providing the appearance of the ordered spatially patterns with the increase of the additive noise and its further destruction. The appearance of the ordered state is a manifestation of the phase transition, so one should understand which factors lead to this transition. To do this, let us follow the argumentation suggested in [6] to give an explanation of the phase transition induced by the multiplicative noise but now influenced by the additive noise.

For a single element of the lattice the time evolution of the first moment is given simply by the drift part in the Fokker-Planck operator, which reads (Stratonovich case)

$$\langle \dot{x} \rangle = \langle f(x) \rangle + \frac{\sigma_\xi^2}{2} \langle g(x) g'(x) \rangle. \tag{15}$$

As it was argued in [6], the evolution over short times of an initial δ function is well approximated by a Gaussian whose extremum obeys

$$\dot{\bar{x}} = f(\bar{x}) + \frac{\sigma_\xi^2}{2} g(\bar{x}) g'(\bar{x}). \tag{16}$$

Here $\bar{x} = \langle x \rangle$ is the maximum of the probability, which is the average value in this approximation. For this dynamics one is able to introduce a potential $U(x) = U_0(x) + U_{\text{noise}} = -\int f(x)dx - \sigma_\xi^2 g^2(x)/4$, where $U_0(x)$ is the unperturbed potential and $U_{\text{noise}} < 0$ describes the action of the noise. In the case under consideration $U_0(x) = x^2(1 + x^2 + x^4/3)/2$, which is monostable with a minimum at $x_0 = 0$.

Let us consider how additive noise modifies the potential $U(x)$. We start with the case of $\sigma_\zeta^2 = 0$ and additive noise is included in the equations through $g(x) = a^2 + x^2$ by the constant a. For small a the potential $U(x)$ remains monostable and there is no possibility of a phase transition in the system. If we increase a, i.e., the intensity of the correlated additive noise, the potential $U(x)$ becomes bistable if $a > a_{\text{crit}} = 1/\sqrt{\sigma_\xi^2}$ [see Fig. 6(a)]. For sufficiently strong coupling this bistability will be the reason for the local ordered regions at short time scales, which coarsen and grow with time. Hence the additive part of the noise in the function g is essential for the occurrence of the nonequilibrium phase transition.

The situation with uncorrelated additive noise ($a=0$ and $\sigma_\zeta^2 \neq 0$) is more complicated. In this case the state $x=0$ always remains stable since the noisy part $U_{\text{noise}}(x) \propto x^4$ [see Fig. 6(b)]. Nevertheless, as it is seen from this figure, for

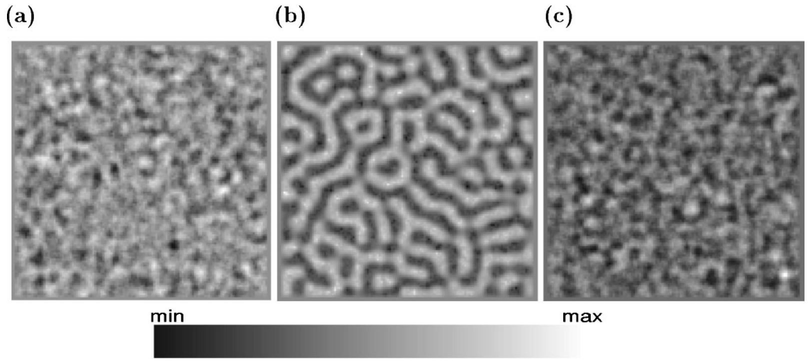

FIG. 5. Snapshots of the field for in the case of the uncorrelated additive noise. The parameter σ_ζ^2 is equal to (1) 0.001, (b) 0.7, and (c) 10.0. The remaining parameters are $D=3.5$, $\sigma_\xi^2=13$, $a=0$, and $\Delta t=10^{-7}$. (max,min) values are $(0.0072,-0.0075)$, $(7.14,-6.33)$, and $(1.07,-0.61)$.

large enough intensity σ_ξ^2, in addition to the stable state $x=0$, the potential $U(x)$ has two minima more, precisely if $\sigma_\xi^2>4$. Therefore, in this case the phase transition is a result of hard excitation and requires independent additive noise. Sufficiently large additive noise causes escapes from the central minimum and the system does not return if the new minimal states are lower than the central one. This argumentation can be considered as an intuitive explanation of the observed noise-induced phase transition by uncorrelated additive noise.

Another interesting finding to be mentioned is the relation between phenomena discussed and the well-known problem of stochastic resonance (SR). Namely, we trace the parallels between the nonmonotonic behavior of the signal to noise ratio (SNR) in SR phenomena and the reentrant phase transitions dependent on the additive noise.

Let us consider possible reasons for this similarity. For that purpose we reformulate the process of ordering in the bistable potential $U(x)$ as a situation typically occurring in SR. The influence of the neighbors supplied by the coupling serves as a driving force for the single system in the lattice with a bistable potential. Under this influence every single system is trying to obey the rules of the whole system, for example, to choose the proper minimum of a potential. Accordance with stochastic resonance becomes evident since this information is best transmitted to the single system if the intensity of an additive noise is optimally selected. For smaller and larger values of noise intensity the ordering process is not effective as in stochastic resonance. As a result and quite analogously to the shape of the SNR, the maximum of the structure function behaves nonmonotonically dependently on the parameter a. The similarities are obviously bounded since in SR the input is independent from the reaction of the system. In our case it differs due to the mutual interaction between the elements of the lattice. It determines the structure of the output, which plays the role of the input for another element.

VI. CONCLUSION

In conclusion, we have shown by the example of the nonlinear model with coupling term similar to that of *Swift and Hohenberg* that an increase of the additive noise may surprisingly induce ordered spatial patterns. The reason is the reentrant phase transition caused by the additive noise. The further increase of the additive noise destroys these structures. In both limiting cases of the correlation between additive and multiplicative noise the pictures are similar but the origins differ. We stress that this phase transition is possible only in the presence of multiplicative noise. As we have discussed, we interpret the phenomenon observed as a coop-

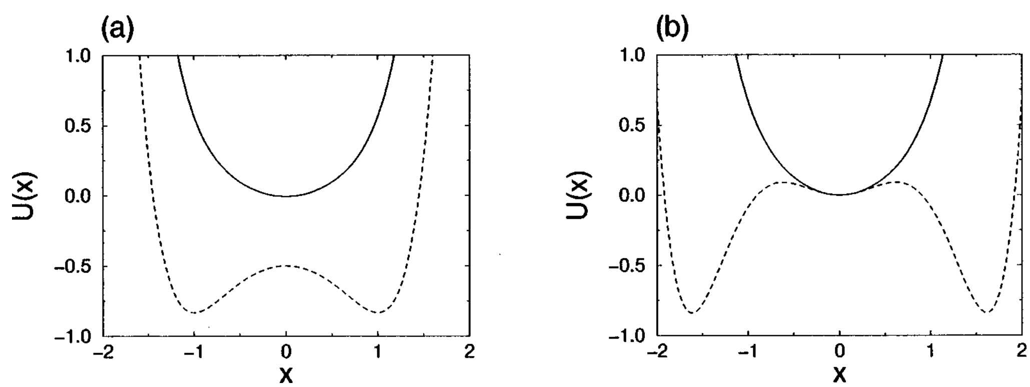

FIG. 6. Potential for the short time evolution of the average value $\langle x(t)\rangle$. (a) $\sigma_\xi^2=2$: solid line, $a^2=0.1$; dashed line, $a^2=1.0$. (b) $a=0$: solid line, $\sigma_\xi^2=2$; dashed line, $\sigma_\xi^2=5$. In case (a) the short time behavior can be described by the bistable potential if the constant a is sufficiently large. In case (b) the situation is more complicated: the state x_0 remains stable, but large enough additive noise can force a system to leave the zero state and form a mean field.

erative work of a noise-induced phase transition and ordering process with an optimal value of the additive noise. From this point of view the phenomena observed can be understood as a mixture of the phase transition induced by the multiplicative noise and processes that have similarities to features of stochastic resonance.

ACKNOWLEDGMENTS

It is a pleasure to thank P. Landa, C. Van den Broeck, J.M.R. Parrondo, J. Buceta, and J. Freund for useful discussions. A.Z. acknowledges support from the DFG (Grant No. SCHI 354/5-1).

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