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Influence of additive noise on noise-induced phase transitions in nonlinear chains

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Abstract—The influence of additive noise on phase transitions in nonlinear chains induced by multiplicative noise is studied. These chains can be considered as finite difference approximations of nonlinear partial differential equations of diffusion type. The phase transitions considered manifest themselves in the disruption of the symmetry and the birth of new equilibrium states. In the so-called mean field approximation, it is shown that additive noise significantly shifts the boundaries of the phase transition. © 1998 Elsevier Science Ltd. All rights reserved.

It was shown in [1–3] that, in certain nonlinear chains and lattices effected by multiplicative noise, nonequilibrium phase transitions of the second kind can occur. These phase transitions manifest themselves in the disruption of the symmetry and the birth of new equilibrium states. The aim of our paper is to study the influence of additive noise on the phase transitions indicated.

In the simplest case, the chain considered is described by the equation

$$\dot{x}_i = f(x_i) + g(x_i)\xi_i(t) - \frac{D}{2}(2x_i - x_{i+1} - x_{i-1}) + \zeta_i(t), \tag{1}$$

where i is a number of the chain cell, $f(x_i)$ and $g(x_i)$ are certain nonlinear functions, $\xi_i(t)$ and $\zeta_i(t)$ are sufficiently wide-band random processes noncorrelated to one another. We note that equation (1) can serve as a finite difference model of a certain nonlinear diffusion equation with noise sources.

The Fokker–Plank equation corresponding to equation (1) is ([4])

$$\frac{\partial \tilde{w}}{\partial t} = -\sum_i \frac{\partial}{\partial x_i}\left((f(x_i) - \frac{D}{2}(2x_i - x_{i+1} - x_{i-1}))\tilde{w} - \frac{\kappa_\xi}{4}\left(\frac{\partial}{\partial x_i}(g^2(x_i)\tilde{w}) + g^2(x_i)\frac{\partial \tilde{w}}{\partial x_i}\right) - \frac{\kappa_\zeta}{2}\frac{\partial \tilde{w}}{\partial x_i}\right), \tag{2}$$

where $\tilde{w}$ is multi-dimensional probability density, κ_ξ and κ_ζ are spectral densities of the processes $\xi(t)$ and $\zeta(t)$ respectively, at the zeroth frequency. Upon integrating equation (2) over all variables with the exception of x_i, we obtain the equation for one-dimensional probability density w for the variable x_i;

$$\frac{\partial w}{\partial t} = -\frac{\partial}{\partial x_i}\left((f(x_i) - D(x_i - M(x_i))w - \frac{\kappa_\xi}{4}\left(\frac{\partial}{\partial x_i}\left(g^2(x_i)w\right) + g^2(x_i)\frac{\partial w}{\partial x_i}\right) - \frac{\kappa_\zeta}{2}\frac{\partial w}{\partial x_i}\right) \tag{3}$$

where $M(x_i) = \langle x_j|x_i\rangle = \int_{\infty}^{\infty} x_j p(x_j|x_i)\mathrm{d}x_j$ is the average of the variable x_j for a fixed value of the variable x_i (conditional average), $p(x_j|x_i)$ is the conditional probability density. In the so-called mean field approximation [1], in place of the conditional average $M(x_i)$, the ordinary average

$$m = \langle x_i\rangle = \int_{-\infty}^{\infty} x_i w_{st}(x_i)\mathrm{d}x_i \tag{4}$$

is substituted into (3). In so doing, a steady-state solution of equation (3) is [4]

$$w_{st}(x) = \frac{C(m)}{\sqrt{\kappa_\xi g^2(x) + \kappa_\zeta}} \exp\left(2\int_0^x \frac{f(y) - D(y-m)}{\kappa_\xi g^2(y) + \kappa_\zeta}\mathrm{d}y\right), \tag{5}$$

where $x \equiv x_i$, $C(m)$ is the normalization constant determined by the following expression:

$$C^{-1}(m) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\kappa_\xi g^2(x) + \kappa_\zeta}} \exp\left(2\int_0^x \frac{f(y) - D(y-m)}{\kappa_\xi g^2(y) + \kappa_\zeta}\mathrm{d}y\right)\mathrm{d}x. \tag{6}$$

Taking into account (4) and (5) we obtain the equation for m:

$$m = F(m), \tag{7}$$

where

$$F(m) = C(m)\int_{-\infty}^{\infty} \frac{x}{\sqrt{\kappa_\xi g^2(x) + \kappa_\zeta}} \exp\left(2\int_0^x \frac{f(y) - D(y-m)}{\kappa_\xi g^2(y) + \kappa_\zeta}\mathrm{d}y\right)\mathrm{d}x. \tag{8}$$

It follows from (8) that $F(m)$ is an odd function of m. Therefore, equation (7) must necessarily have a solution $m = 0$. If $D = 0$ then this solution is a single one. But if $D > 0$, then, from a certain value of the noise intensity κ_ξ onwards, two other solutions, which are nonzero ($m = \pm m_0$), can appear.

The computation of equation (7) shows that the condition for the existence of nonzero solutions of equation (7) is

$$\left|\frac{\mathrm{d}F}{\mathrm{d}m}\right|_{m=0} \geq 1. \tag{9}$$

In [1] the functions $f(x)$ and $g(x)$ were given in the form

$$f(x) = -x(1+x^2)^2, \quad g(x) = 1 + x^2, \tag{10}$$

and the noise $\zeta(t)$ was absent. Nevertheless, as evident from equation (1), the specification of the function $g(x)$ in the form (10) implies that not only multiplicative noise but additive one acts on the system under consideration as well. We note that in this case additive noise is correlated with multiplicative one.

In [1], the boundary of the noise-induced phase transition on the plane of the parameters κ_ξ and D was found numerically from the condition (9), in view of the expressions (8) and (6). The corresponding values of m_0 versus κ_ξ were obtained from equation (7) for $D = 20$. The immediate determination of the values of m_0 from numerical simulation of equation (1) showed that these values are close to that calculated in the mean field approximation only for very small values of the noise intensity κ_ξ ($\kappa_\xi < 4$). For larger κ_ξ, there is considerable disagreement between these results ([1]). The latter can be attributable to either the fact that, by virtue of a finite correlation time of the noise, the Fokker–Plank equation becomes invalid for moderately large noise intensities ([4]), or breaking down the mean field approximation for such noise intensities.

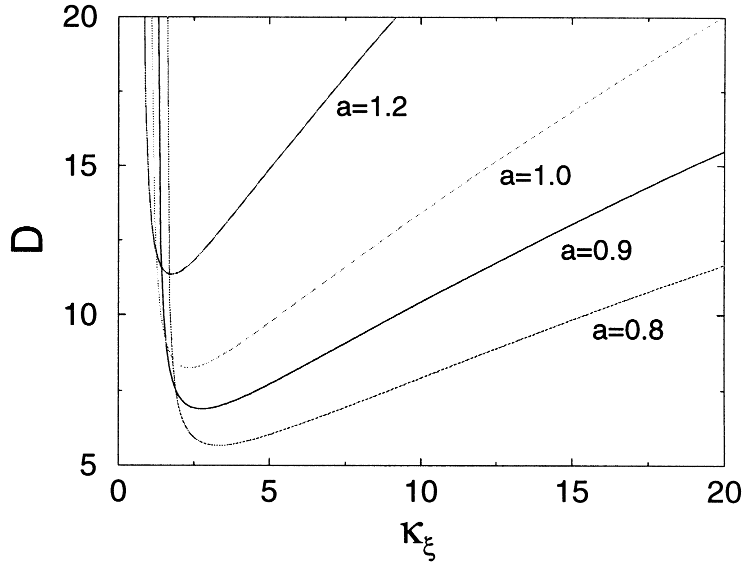

Fig. 1. The boundaries of the noise-induced phase transition on the plane of the parameters κ_ξ and D for different values of a (shown in the figure). The variation of a causes significant shift of the boundary of the phase transiton.

To follow the influence of additive noise, we first set the function $g(x)$ in the form of $g(x) = a^2 + x^2$, where the parameter a determines the relation between intensities of additive and multiplicative noise components. The boundaries of the noise-induced phase transition on the plane of the parameters κ_ξ and D, for $\zeta = 0$ and different values of a, are shown in Fig. 1. It follows, from this figure, that as a decreases, the boundary of the phase transitions on the plane κ_ξ, D is significantly dropped and right shifted. Corresponding dependencies of m_0 on κ_ξ for $D = 15$ are given in Fig. 2.

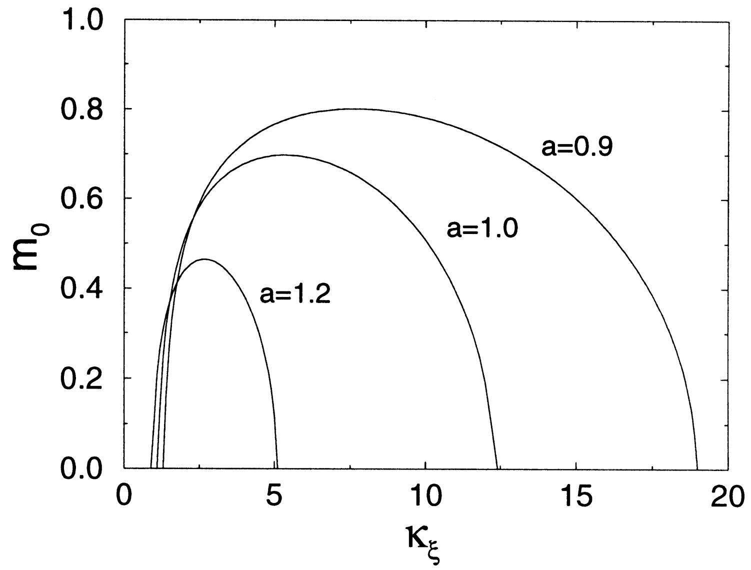

Fig. 2. The plot of m_0 versus κ_ξ for $D = 15$ and different values of a (shown in the figure).

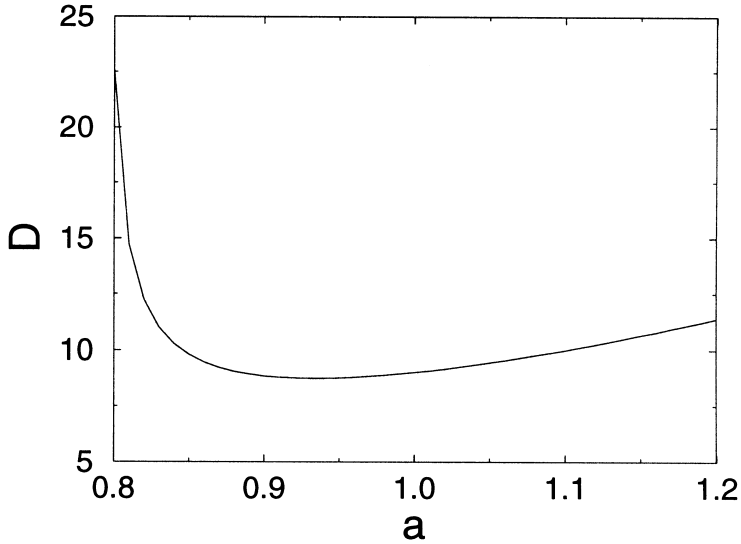

Fig. 3. The boundaries of the noise-induced phase transition on the plane of the parameters a and D for $\kappa_\xi = 1.6$.

As may be seen from Fig. 1, additive noise plays a great role for comparatively small values of the noise intensity κ_ξ, namely, in the domain where the boundary value of D decreases as κ_ξ increases. As a increases, the boundary value of D first drastically decreases and then slightly increases. This is illustrated by Fig. 3, in which the dependence of the boundary value of D on a is shown for a fixed value of κ_ξ ($\kappa_\xi = 1.6$). We see that the influence of the additive noise component is very similar to the multiplicative noise: for fixed value of the diffusion factor D, the increase of the additive noise intensity first causes the phase transition in one direction and then in the opposite one. This effect can also be illustrated by the dependendence of m_0 on a for a fixed value of D ($D = 10$), see Fig. 4.

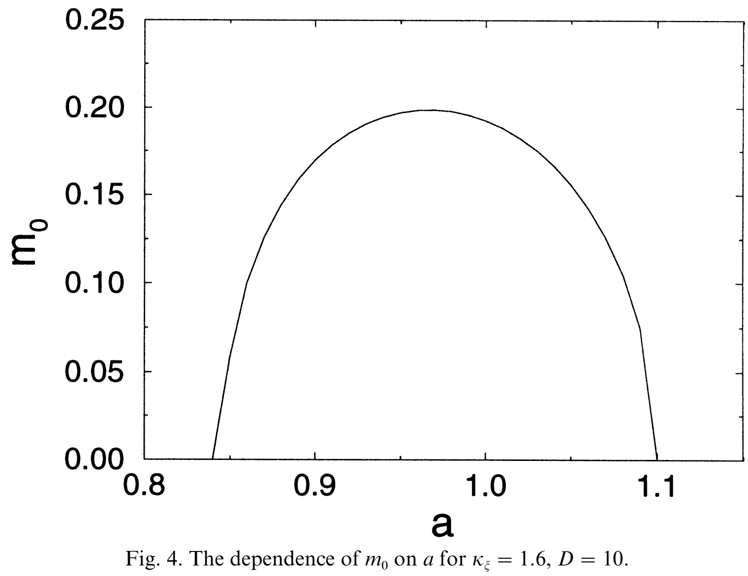

Fig. 4. The dependence of m_0 on a for $\kappa_\xi = 1.6$, $D = 10$.

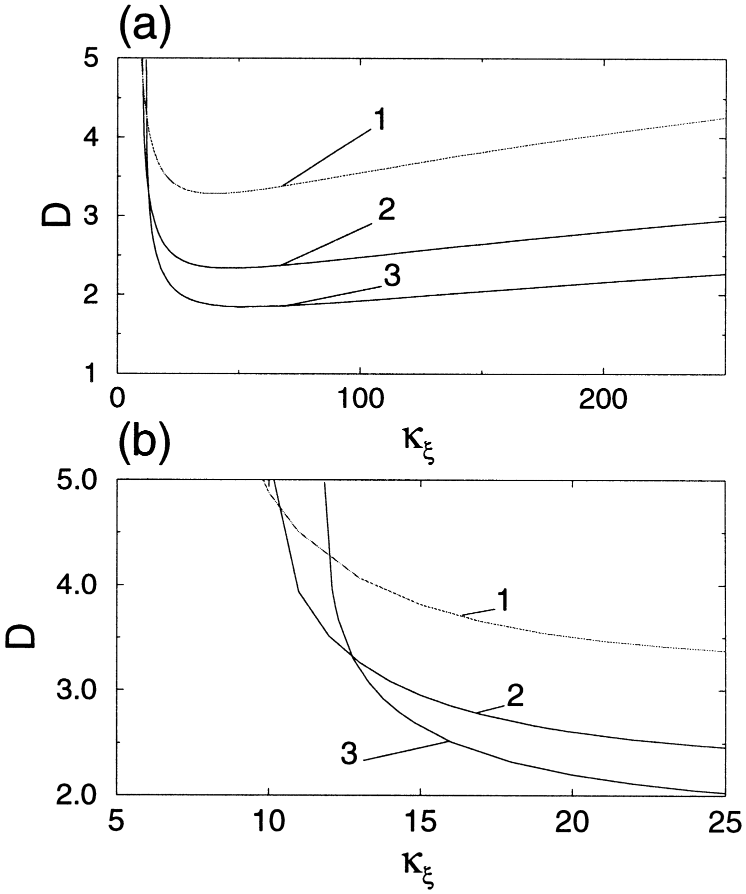

Fig. 5. (*a*,*b*) The boundaries of the noise-induced phase transition on the plane of the parameters κ_ξ and D for $a = 0$ and different values of κ_ζ (1: $\kappa_\zeta = 1$, 2: $\kappa_\zeta = 0.5$, 3: $\kappa_\zeta = 0.3$).

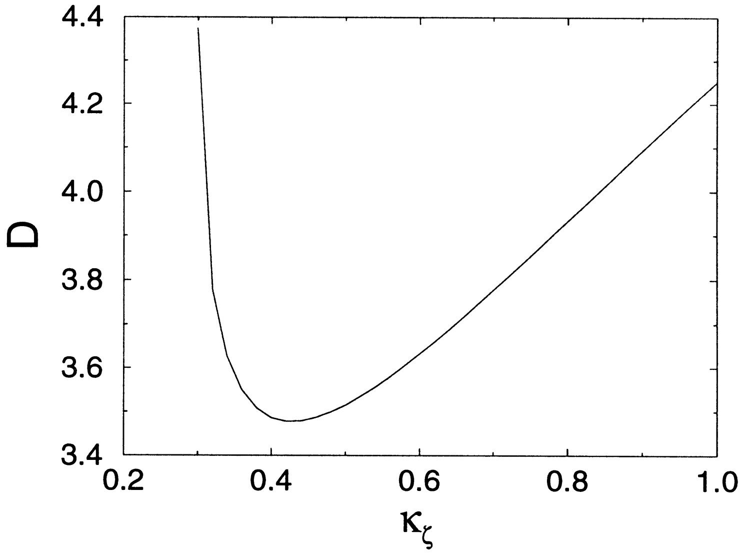

Fig. 6. The boundaries of the noise-induced phase transition on the plane of the parameters κ_ζ and D for $a = 0$ and $\kappa_\xi = 12$.

The influence of additive noise that is noncorrelated with multiplicative once is illustrated in Figs 5 and 6. These figures are obtained by computation of equation (7) in view of (8) for $a = 0$, i.e., $g(x) = x^2$, and different values of κ_ζ. It can be seen that, in this case with the increase of κ_ζ, the boundary of the phase transition is significantly shifted to smaller values of D and greater values of κ_ξ. The dependence of the boundary value of D on κ_ξ is similar to the case with correlated additive noise (see Fig. 6).

So, we have investigated the influence of additive noise on the noise-induced phase transitions in nonlinear chains. We have found a strong influence of additive noise which manifests itself in the shift of the boundaries of the phase transition. We considered two limiting cases of correlation between the additive and multiplicative noise. In both cases, in the presence of multiplicative noise the additive noise can even induce the phase transition.

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