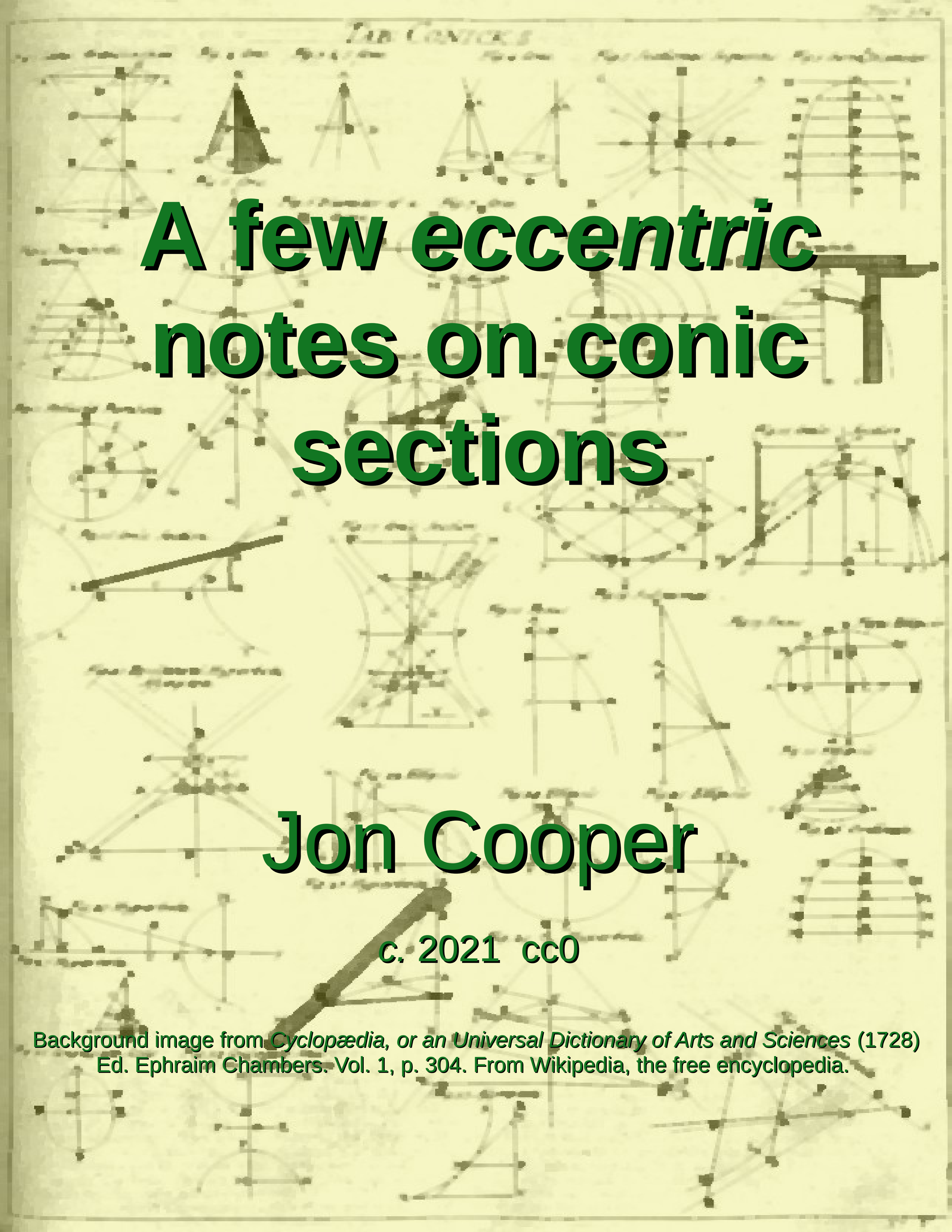


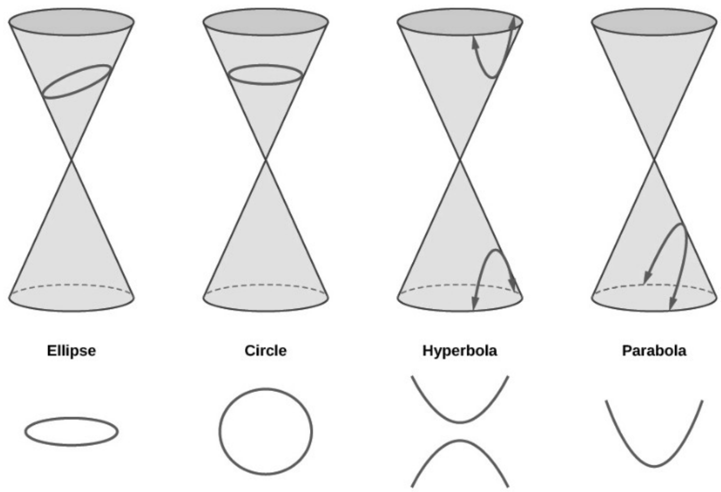
A few *eccentric* notes on conic sections

Jon Cooper

Background image from *Cyclopædia, or an Universal Dictionary of Arts and Sciences* (1728) Ed. Ephraim Chambers. Vol. 1, p. 304. From Wikipedia, the free encyclopedia.

Introduction

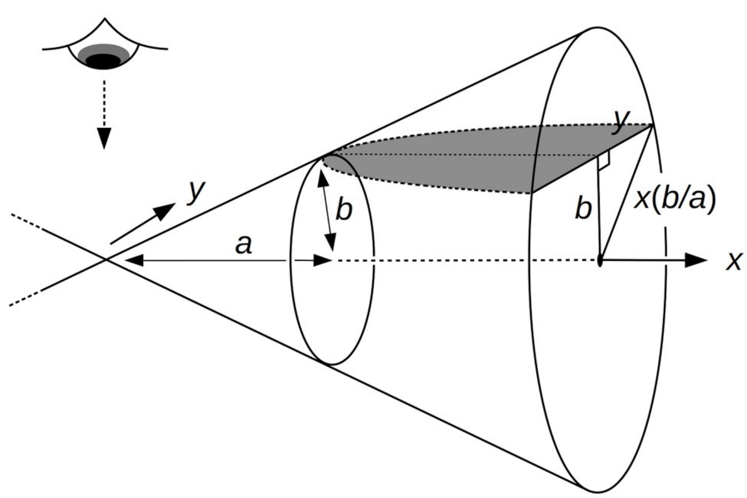
A picture (like the one below adapted from this free textbook [1]) is worth a thousand words.



My interest in this subject was triggered by this excellent article [2] which, although a bit too long and complicated for me to follow, gave me the idea of trying to derive the equations of the conic sections from just that – *plane sections of a cone!* The fairly simple approach here [3] suggested it might work!

The proofs

Start with a right-circular double conic (*i.e.* the central axis is perpendicular to the base of each cone). Assuming that a plane which crosses both halves of the conic, is a **hyperbola**, proof of the equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is easiest if we make the intersecting plane parallel with the cone axis, as shown below.



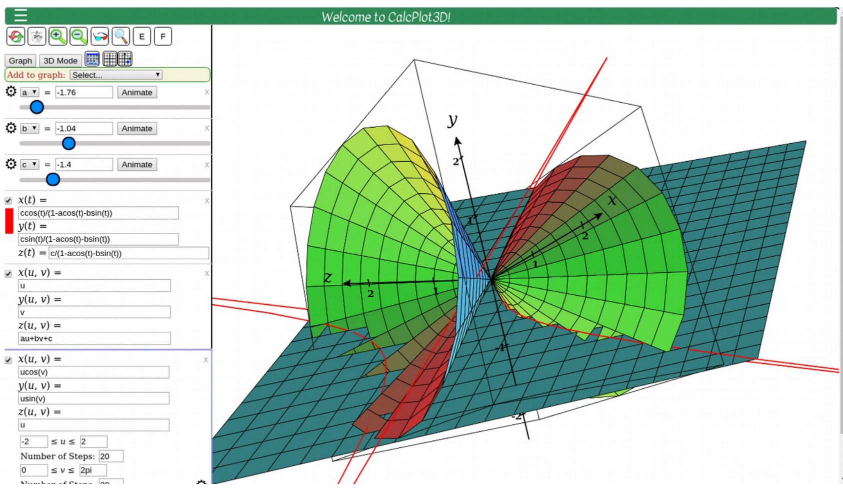
By similar triangles we can see that the circle at the base of the cone has radius *xb/a*. From the right-angled triangle drawn in this circle we get:

$$b^2 + y^2 = \left(x\frac{b}{a}\right)^2$$

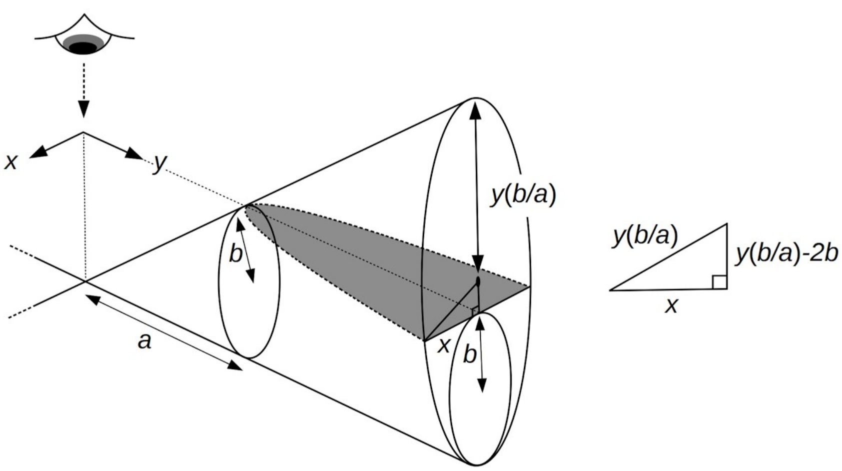
$$\therefore 1 + \frac{y^2}{b^2} = \frac{x^2}{a^2}$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

The angle between the asymptotes of the hyperbola is the angle at the tip of the cone, *i.e.* 2 arctan(*b/a*), and when this is 90°, *i.e.* *a* = *b*, the hyperbola is said to be rectangular, *e.g.* in the Michaelis-Menten equation. The plane will also intersect the left hand cone and give a symmetrical curve. Indeed, when the plane is not parallel with the cone axis, both halves of the hyperbola are still symmetrical, as shown roughly below using CalcPlot3D.



Proof of the general form of the **parabola**. When the plane intersects the double conic parallel with the surface it will only cross one of the cones and this defines a parabola. In the figure below, note that the axes are swapped and oriented such that *y* and *a* are parallel with the cone surface.



Considering the right-angled triangle on the base of the cone (redrawn on the right) gives:

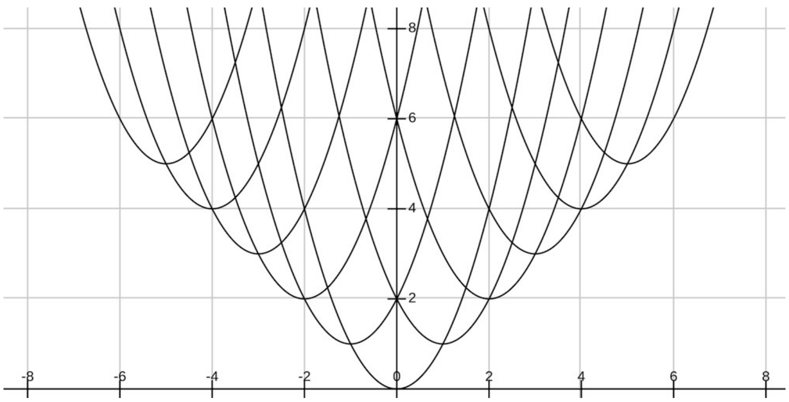
$$x^2 = (y\frac{b}{a})^2 - (y\frac{b}{a} - 2b)^2$$

$$= (y\frac{b}{a})^2 - ((y\frac{b}{a})^2 - 4b(y\frac{b}{a}) + 4b^2)$$

$$= 4\frac{b^2}{a}y - 4b^2$$

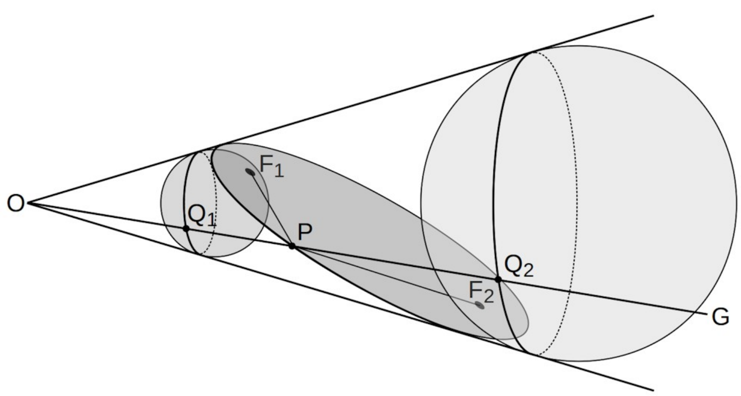
$$\therefore y = (\frac{a}{4b^2})x^2 + a$$

and the form of a simple parabola $y = Ax^2$ where *A* is constant, is proved. The general form of a simple parabola is, of course $y = Ax^2 + Bx + C$, but the effect of the extra terms (*Bx* and *C*) is simply to shift the parabola to different locations, as shown in the fooplot.com graph below.



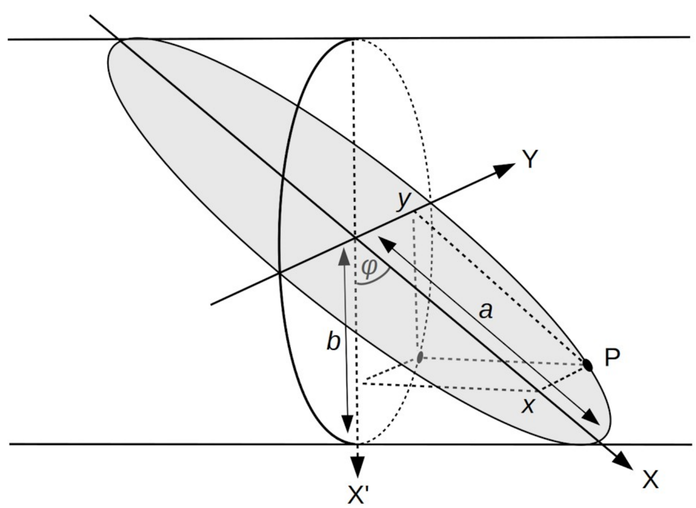
Note that this derivation has been for a vertical parabola, whereas most sources e.g. this excellent one [4] , derive the formula for a horizontal one *i.e.* $y^2 = 4ax$.

When the plane crosses only one of the cones, forming a closed curve, this is an **ellipse**. In the figure below (adapted from here [5]), line OG is the *generator* of a cone which is occupied by two *Dandelin spheres* of different size, that are not touching each other. Each sphere makes a circular line of contact with the inside of the cone and these two circles of contact are perpendicular to the cone axis (not drawn) and are therefore parallel with each other.



The distance Q_1Q_2 is therefore a constant, regardless of where the generator (OG) lies on the surface of the cone. The figure above shows a dark grey plane, passing through the cone, which touches the spheres at points F_1 and F_2 and the generator at P. Lines PF_1 and PQ_1 are therefore tangents to the sphere on the left and lines PF_2 and PQ_2 are tangential to the sphere on the right, meaning that $PF_1 = PQ_1$ and $PF_2 = PQ_2$. Hence, $PF_1 + PF_2 = PQ_1 + PQ_2$. Since, $PQ_1 + PQ_2$ is a constant, regardless of where P lies on the ellipse, $PF_1 + PF_2$ must also be constant. This is the classic definition of the ellipse, allowing it to be drawn by fixing the ends of a piece of string to the page with pins and sliding the tip of a pencil inside the enclosed loop. F_1 and F_2 are therefore the familiar foci of the ellipse, *i.e.* the points on the dark grey plane where the drawing pins would have to be placed in order for a pencil at P to trace out the ellipse. When the pencil is at either end of the long axis, its distance from the nearest pin will be the same on each side of the ellipse. Hence, the foci must be equidistant from the centre of the ellipse.

Deriving the equation of the ellipse from a cone (e.g. here [6]) is too hard so we'll imagine that the cone is infinitely long *i.e.* it degenerates to a cyclinder. A plane intersecting a cylinder at right angles to its axis will give a circular cross-section. At any other angle, the intersection will be elongated. We can imagine two Dandelin spheres inside the cylinder, although, of course, they would have to be of equal size this time. We can have an intersecting plane which touches both spheres, making the treatment essentially the same as above, but it is easier just to do the trigonometry, as below, without Dandelin spheres.

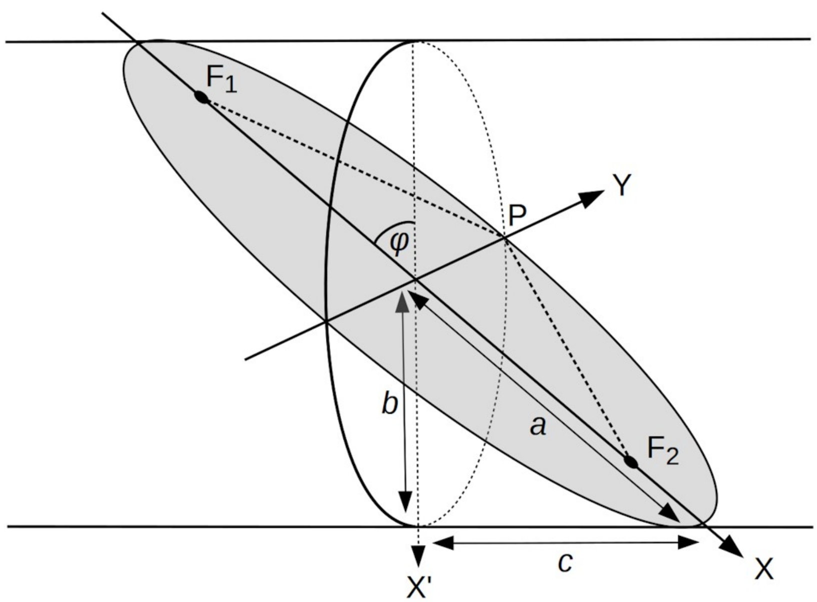


Consider a circle (shown in the vertical direction above) which is formed by a plane crossing a cylinder at right-angles to its long axis. If we tilt this plane through angle φ around the Y-axis which, as shown in the figure, is also perpendicular to the cylinder axis, we get the grey shaded ellipse and $\sin\varphi$ is referred to as its eccentricity (e). The *semi-major* and *semi-minor* axes of the ellipse are a and b, respectively, and b is also radius of the original circle. Point P has coordinates (x, y) with respect to the X- and Y-axes in the tilted grey plane. The X-axis of the original circle is shown as X'. The projection of P onto the plane of the original circle has coordinates ($x\cos\varphi$, y) with respect to the X'- and Y-axes and since $\cos\varphi = b/a$, the coordinates become $\left(x\frac{b}{a}, y\right)$ and we can use Pythagoras as follows:

$(x\frac{b}{a})^2 + y^2 = b^2$ giving $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ as the equation for the ellipse.

This is not so easy to do with a plane crossing a cone at right-angles because the centre of the ellipse that is formed by tilting the plane moves away from the centre of the circle [6]. Right, so we have an equation for an ellipse, but is it consistent with what we derived from the Dandelin spheres in a cone, *i.e.* that the sum of the distances of P from its two foci must be a constant? Given that we could have two Dandelin spheres of equal size in a cylinder and derive the same result from a plane touching both, its looking pretty promising.

But, to be sure lets go back to the cylinder. If we were to place the point P in the above figure exactly on the positive side of the X-axis (i.e. at x = a and y = 0), if this really is an ellipse then, since the foci must be equidistant from the centre, PF_1+PF_2 must be 2a. If instead we place P exactly on the positive Y-axis (as shown below) then x = 0 and y = b. Since $PF_1 = PF_2$ and PF_1+PF_2 = 2a then $PF_1 = PF_2 = a$. Hence, the triangle formed by P, F_2 and the origin is identical with that formed by X, X' and the origin. Hence, the distance of each focus from the origin is given by the following equivalent expressions: $\sqrt{(a^2-b^2)}$, $a$sinφ or ae, and is shown as c below. This distance, c, is called the *linear eccentricity*.



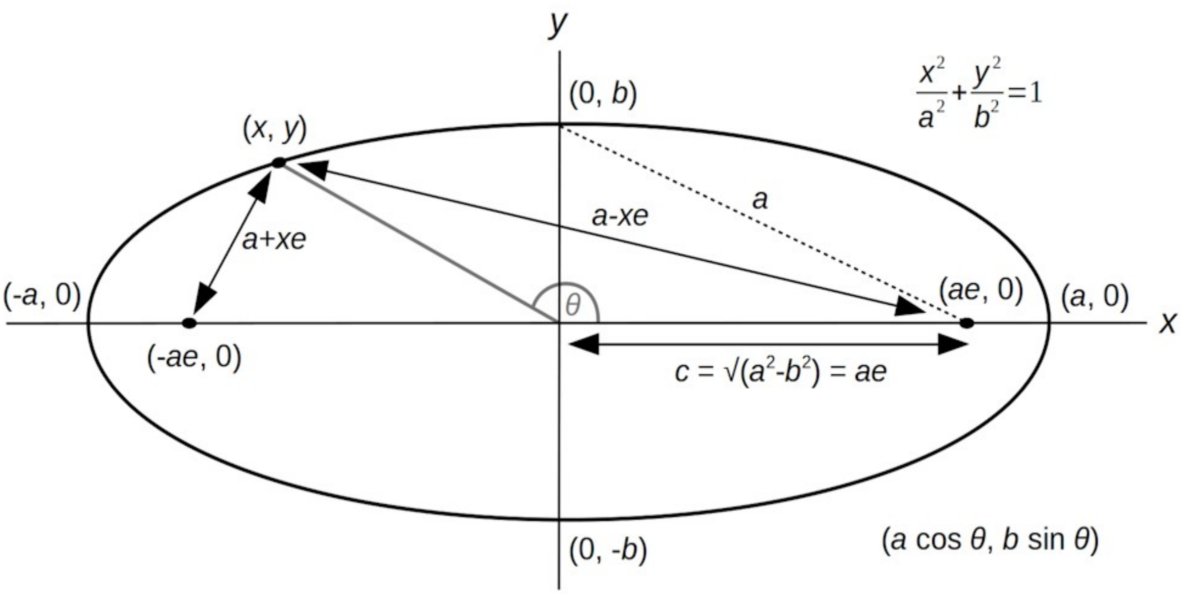
So, if we take the shape formed by intersecting a plane with a cylinder and consider the foci to be at ±ae on the semi-major axis, then we have a shape that is identical in form to the ellipse formed by the intersection of a plane with a cone, I thik.

In the figure above we placed P at the special position (0, b) and showed that the distance from each focus is a. When P is at the general position (x, y), its distance from each of the foci is a bit harder to work out, but not too bad. If we consider F_2 which is at position (c, 0), the square of its distance from P is given by $PF_2^2 = (c-x)^2 + y^2$ where $c = ae = \sqrt{(a^2-b^2)}$. Given that the equation for the ellipse can be rearranged as $y^2 = b^2(1-x^2/a^2)$ we get:

$$PF_2^2=(c-x)^2+b^2(1-\frac{x^2}{a^2}) \quad =c^2-2cx+x^2+b^2-b^2\frac{x^2}{a^2}=a^2-b^2-2cx+x^2+b^2-b^2\frac{x^2}{a^2} \quad =a^2-2cx+x^2-b^2\frac{x^2}{a^2}$$
$$=a^2-2cx+x^2(1-\frac{b^2}{a^2}) \quad =a^2-2cx+x^2(\frac{a^2-b^2}{a^2})=a^2-2cx+x^2e^2 \quad =a^2-2aex+x^2e^2 \quad =(a-xe)^2$$

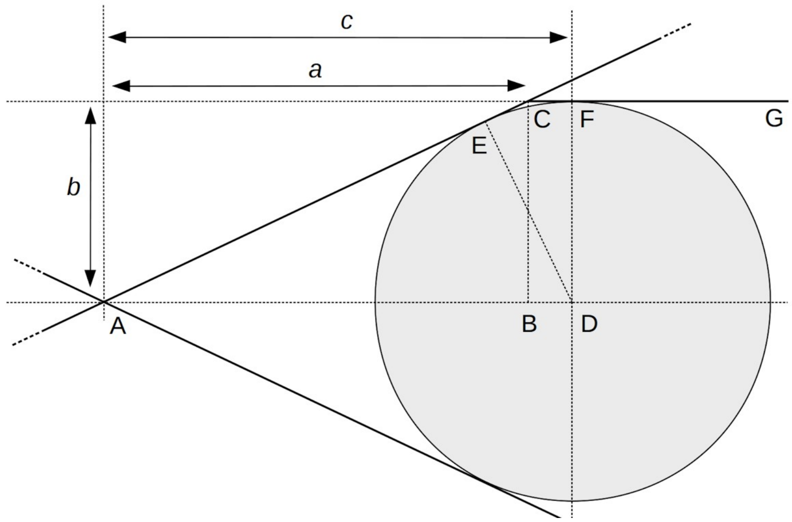
Hence, $PF_2 = a - xe$ and since $PF_1^2=(c+x)^2+y^2$ we can see that $PF_1 = a + xe$, *i.e.* the sum of the distances of P from the foci is, as expected, $2a$. Hence, the formula for the ellipse derived from the cylindrical section is consistent with the conic.

A summary of the ellipse formulae is given below.



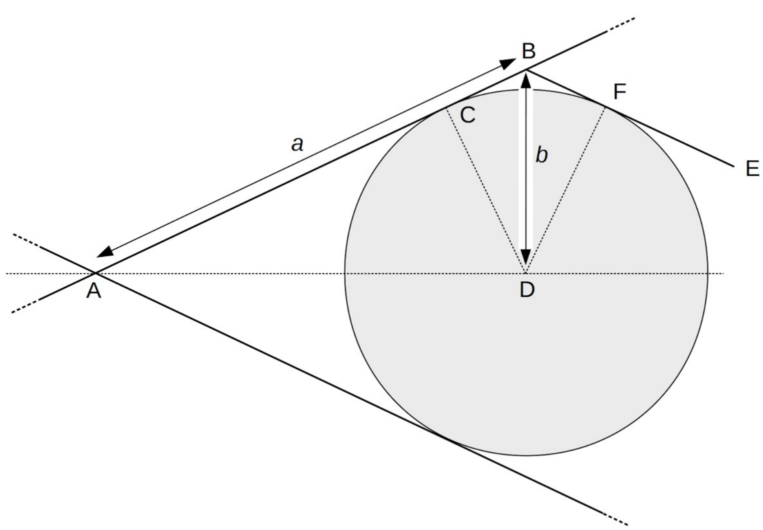
Eccentricity is defined as the ratio of *c/a i.e.* the distance from centre to focus divided by the distance from centre to vertex. For a circle the centre is the focus so $e = 0$. For the ellipse, we have derived the formula of $e=\sqrt{(a^2-b^2)}/a$ and this will be in the range $0 < e < 1$ since when the eccentricity approaches 1, the ellipse becomes a parabola. One advantage of the conic visualisation of the ellipse over the cylindrical one is that we can see that if the intersecting plane becomes very tilted, *i.e.* φ approaches 90°, the ellipse becomes infinitely long and open on one side but one vertex and focus will remain in the finite realm.

For the hyperbola, which is basically two open curves with their vertices on opposite sides of the origin, the distance of each vertex from the centre is *a*, as in the earlier figure, but what is the distance to the focus? Hmmm. We can use a Dandelin sphere in each of the cones of the double conic which touches the plane of the hyperbola. Redrawing the hyperbola figure so that we are looking exactly along the Y-axis with the X-axis horizontal gives the following view of one half of the double-conic.



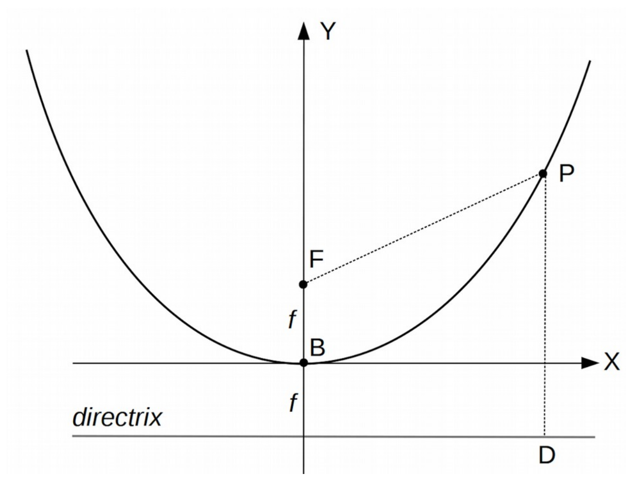
Line CG is the plane of the hyperbola which has a vertex, C, at a horizontal distance of *a* on the X-axis, where the radius of the cone is *b*. A Dandelin sphere with centre at D and touching the plane of the hyperbola at F, the focus, is shown. Since triangles ABC and AED are congruent, the focal distance $AD = AC = c = \sqrt{(a^2 + b^2)}$. Hence for the hyperbola, the eccentricity, *e* or *c/a*, is greater than 1. Note that AB = AE = *a*, *i.e.* the circle of contact between the sphere and the cone is at distance *a* from the tip of the cone, as measured on the surface of the cone. Note also that we are relying on the assumption that wherever the Dandelin sphere touches the plane containing the curve is the focus of the curve. We need to look more at what a focus actually is and will do this later on.

Now to the parabola and first we need to find where the focus is, which can be done using a similar treatment to the above. In the figure below, line BE is a sideways-on view of the plane containing the parabola which has its vertex at B and focus at F. The line through AB is the generator of the cone and DB is its radius at the vertex of the parabola. Since triangles ABD, ACD, BCD and BDF are all similar, we can see that BF is b^2/a, *i.e.* the focus of the parabola is at this distance from the vertex.



A problem arises with the parabola because it doesn't have a centre from which to measure distances to the vertex and focus. This is because the centre is at infinity *i.e.* it's like the two curves of a hyperbola or a closed ellipse with one of the two halves infinitely far away. So how do we measure the eccentricity without a centre? Well, it seems you can use something which I have been deliberately avoiding till now, namely the *directrix*. Most textbooks start with the directrix, which is just a line, and derive everything about conic sections from it, but to me it seemed better to start with an actual cone and relate everything to that, since these are called conic sections after all! So what is this distractrix all about then? Apparently it is a line from which you measure an orthogonal distance to point P on the curve. The distance of P from the focus is then expressed as a fraction of the distance from the directrix and this is the eccentricity. The textbook definition of a conic section is that this ratio must be constant for any given curve. Fine, but confusing without a diagram.

Consider the first diagram of the parabola from which we derived $y=(ax^2/4b^2)+a$. If, instead, we had placed the origin at the vertex of the parabola, we would have derived the following simpler equation: $y=(ax^2/4b^2)$, and we've just shown that the focal distance of the parabola (*f*) is b^2/a, so we can now say: $y=x^2/(4f)$. This is drawn below and gives an interesting result that when *y = f, x = 2f*, *i.e.* the full horizontal width of the parabola at the focus (referred to as the *latus rectum*) is therefore 4*f*.



With the parabola, the latus rectum running horizontally through F (not shown) and the directrix are parallel with the X-axis and are equidistant from it. The ratio PF/PD should therefore be constant and give us the eccentricity. PD is given by:

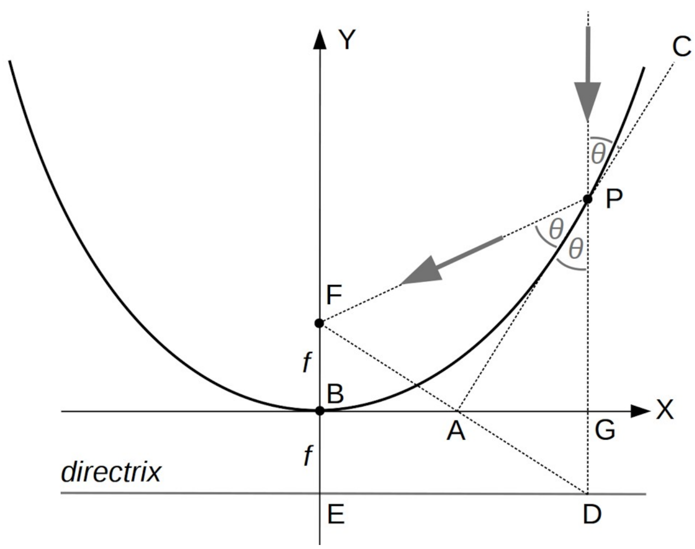
$$PD=y+f=\frac{x^2}{4f}+f$$
$$\therefore PD^2=\left(\frac{x^2}{4f}+f\right)^2$$
$$=\left(\frac{x^2}{4f}\right)^2+2\left(\frac{x^2}{4f}\right)f+f^2$$
$$=\left(\frac{x^2}{4f}\right)^2+\frac{x^2}{2}+f^2$$

PF is determined by Pythagoras as:

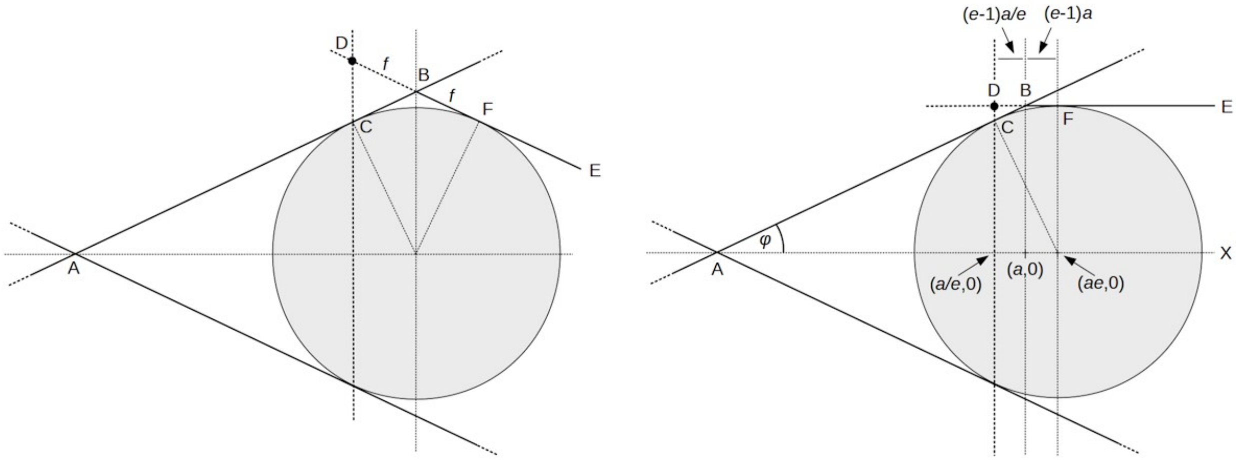
$$PF^2 = (y-f)^2 + x^2 = \left(\frac{x^2}{4f} - f\right)^2 + x^2$$
$$= \left(\frac{x^2}{4f}\right)^2 - 2\left(\frac{x^2}{4f}\right)f + f^2 + x^2$$
$$= \left(\frac{x^2}{4f}\right)^2 - \frac{x^2}{2} + f^2 + x^2$$
$$= \left(\frac{x^2}{4f}\right)^2 + \frac{x^2}{2} + f^2$$

Hence, PF = PD and the ratio PF/PD or eccentricity of the parabola is 1.

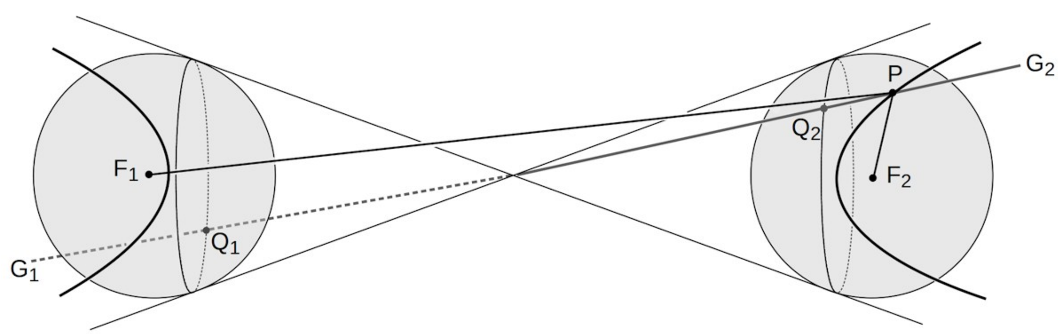
To prove that the focus really is a focus, consider a ray of light travelling in the direction of PD which is being reflected by a parabolic mirror with angle of incidence, θ, equal to the angle of reflection. Given that all of the triangles below are similar and PF = PD, FB = BE = f, *etc.*, we can see that FA = AD, *i.e.* line PA bisects the angle between PF and PD such that line AC behaves as a perfect mirror plane at P. The gradient of AC is PG/AG or ($\frac{x^2}{4f} \div \frac{x}{2}$) or $x/(2f)$ which is the same as the gradiant of the parabola that can be obtained by differentiation. Hence, line AC must be a tangent to the parabola which reflects the ray towards the focus, F.



So, having introduced the directrix, where is it in relation to the Dandelin sphere? This question is addressed in this online essay [7]. The figure on the left below shows that it lies on the intersection of the plane of the circular contact between the cone and the Dandelin sphere (line through DC) and the plane of the parabola (line through DE). Both planes are perpendicular to the page, *i.e.* they appear as lines, and the line of intersection, or directrix, is also perpendicular to the page and is shown as a black dot at D. The figure on the below right shows the situation with the hyperbola where the lengths of DB and BF are not equal. We showed earlier that the focus of the hyperbola is at a distance of (c-a) or $a(e$-1) from the vertex (shown as horizontal line BF). Since CB = BF and DB = CB cosφ , we can see that the distance of the directrix from the vertex, DB = $a(e$-1)a/c or (e-1)a/e.



A simplified Dandelin sphere treatment of the hyperbola is shown for the double conic below which has generator G_1G_2. The plane of the hyperbola is parallel with the cone axis and therefore both Dandelin spheres are the same size. Using the same considerations as for the ellipse we see that $PF_1 = PQ_1$ and $PF_2 = PQ_2$. Point P traces out a hyperbola by swivelling the generator G_1G_2 around the cone axis. The difference $PF_1 - PF_2$, which equals $PQ_1 – PQ_2$, is therefore constant and equal to the distance along the generator between the circles of contact between each cone and its Dandelin sphere. We showed above that these circles are at distance *a* from the tip of each cone, as measured on the surface of the cone, and we can therefore say that $PF_1 – PF_2 = 2a$. This equation for the hyperbola is analogous to the one we derived for the ellipse ($PF_1 + PF_2 = 2a$) using Dandelin spheres. The figure below was adapted from [8][9] and shows the plane of the hyperbola in the plane of the page with the Dandelin spheres below it, for greater simplicity.



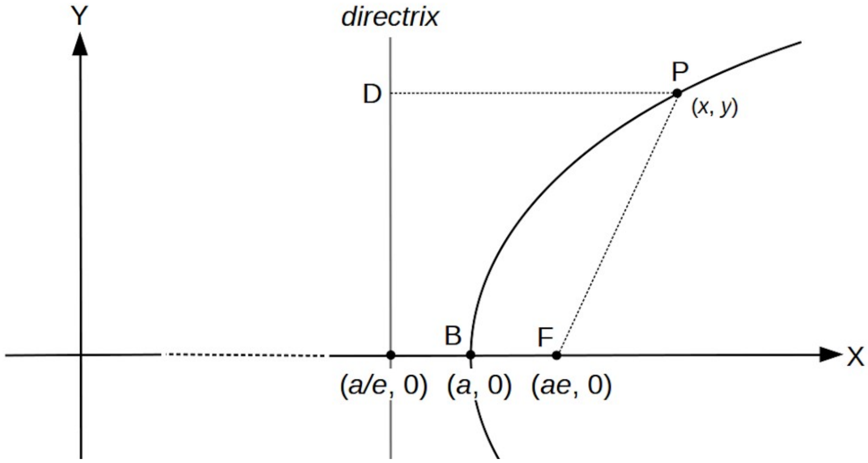
As we did with the ellipse, we can check that the equation $PF_1 – PF_2 = 2a$ agrees with the equation relating the Cartesian *x, y* coordinates. If we consider F_2 which is at position (*c*, 0), the square of its distance from P is given by $PF_2^2=(x-c)^2+y^2$ where $c=ae=\sqrt{(a^2+b^2)}$. Given that the equation for the hyperbola can be rearranged as $y^2=b^2(x^2/a^2-1)$ we get:

$$PF_2^2=(x-c)^2+b^2(\frac{x^2}{a^2}-1) \quad =x^2-2xc+c^2+b^2\frac{x^2}{a^2}-b^2 \quad =x^2(1+\frac{b^2}{a^2})-2xc+c^2-b^2$$

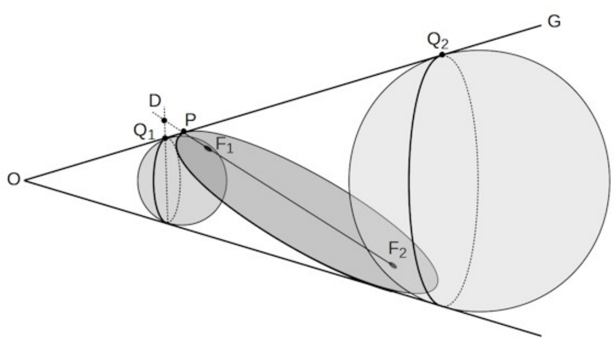
$$=x^2(\frac{a^2+b^2}{a^2})-2xc+a^2 \quad =x^2(\frac{c^2}{a^2})-2xc+a^2 \quad =x^2e^2-2aex+a^2 \quad =(xe-a)^2$$

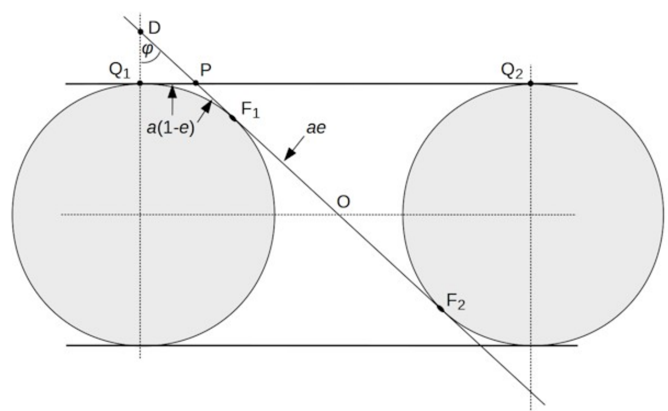
Hence, $PF_2 = xe\text{-}a$ and similarly $PF_1 = xe+a$, *i.e.* the difference between the distances of P from the foci is, as expected, $2a$.

The next thing to check is that the directrix whose position we determined on the right hand side of the figure before the last one agrees with the rule that PF/PD should equal the eccentricity. We can see in the figure below that PD = $x\text{-}a/e$ or $(xe\text{-}a)/e$ and we have just shown that PF is $(xe\text{-}a)$, i.e. PF/PD = e.



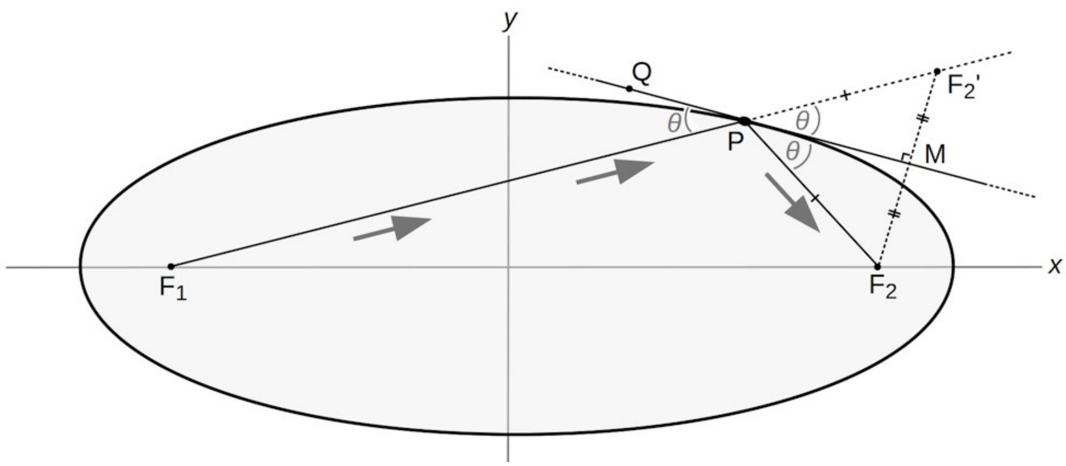
Right, since we're dawdling around Dandelin spheres, where are the directrices of an ellipse? The ellipse formed by two Dandelin spheres in a cone is shown on the left with the generator OG in the fully upright position, *i.e.* P is collinear with the foci, F_1 and F_2. From the ellipse summary diagram we know that PF_1 and PQ_1 are equal to $a(1\text{-}e)$, but the rest of the maths for a cone is too hard, so I swapped back to the cylinder representation with two Dandelin spheres of equal size and P at a vertex of the ellipse, as on the right. This time, we are viewing the plane of the ellipse exactly sideways on so the directrix D is orthogonal to the page. The line DP shows the distance of the directrix from the vertex of the ellipse, which can be determined from the triangle DPQ_1 since $DP = PQ_1 / \sin\varphi$ or $a(1\text{-}e)/e$. The distance of D from the centre O is OP + DP (or $a+a(1\text{-}e)/e$) which works out easily as a/e. Reassuringly, these formulae are the same as those we derived for the hyperbola.





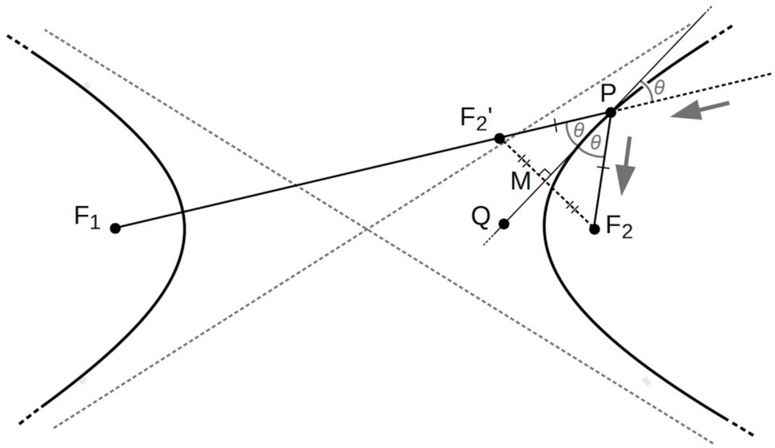
We now know where the directrices are for the ellipse and the hyperbola, but can we show what features the foci have *e.g.* are they really foci as with a parabolic mirror?

The case of an ellipse is shown below and is adapted from a slightly scary one on Stack Exchange [10], but I am not expert enough to fault it. The idea is to extend the line F_1P by the length of F_2P and there place the point F_2'. If we then place the point M so as to bisect $F_2'F_2$ and draw a line through P and M, then with all the similar triangles we have something which looks very much like a tangent, with angle of incidence (θ) equalling the angle of reflection, *etc.*, but is it really a tangent? The idea is that it only touches the ellipse at one point (P) so it must be a tangent because any other line through P would have to cross the ellipse somewhere else, too. But can we be sure that this line does only touch the ellipse once? The answer to that is that if we take any other point on the line, represented by Q, then since it is outside the grey shaded area of the ellipse, we can, almost Venn-diagram style, say that $F_1Q + F_2Q$ must always be bigger than $2a$ and therefore Q can never obey the sum of the distances to the foci rule. Bit hand-wavy, but saves doing pages of algebra and it seems to confirm that waves emitted at F_1 (shown as thick grey arrows) would all be reflected towards F_2 and *vice versa*, examples being elliptical rooms, allowing someone at one end to overhear people whispering at the other.



On re-reading the proof, the clever bit is that $F_2P = F_2'P$ *i.e.* the length of the line from F_1 to F_2' is $2a$. Considering triangle F_1QF_2', the sum of F_1Q and $F_2'Q$ must be greater than $2a$ (or it wouldn't be a triangle, aka the *triangle inequality*) and since $F_2'Q = F_2Q$, it is easy to see that $F_1Q + F_2Q > 2a$, *i.e.* Q must always be outside the ellipse, so the line PM must be a tangent.

Using the same notation, I adapted this proof for the hyperbola which confirms its focus property. The main difference is that the line F_1P is extended backwards by the distance F_2P and the line PM is drawn such that it bisects F_2F_2'. Since $F_1P - F_2P = 2a = F_1P - PF_2'$ we can see that F_1F_2' must be $2a$. Considering triangle $F_1F_2'Q$ the triangle inequality dictates that $F_1Q < F_1F_2' + F_2'Q$ or $F_1Q - F_2'Q < F_1F_2'$. Hence $F_1Q - F_2Q < 2a$ and Q therefore cannot lie on the hyperbola. Note also that the tangent can never cross the other half of the hyperbola since its slope is always limited by the asymptote (grey dashed lines). Wikipedia has the same proof, so it can't be wildly wrong, and confirms that hyperbolic mirrors have a wider field of view than parabolic ones, as the diagram below suggests.



Gradients of ellipse and hyperbola

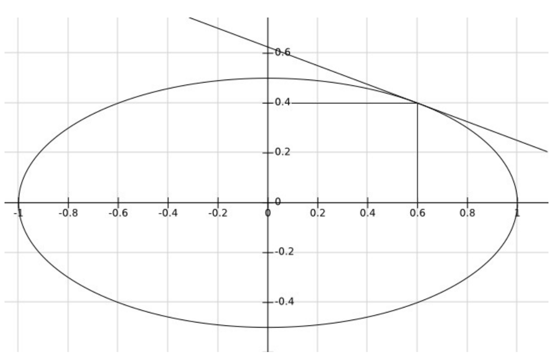
The equation of the ellipse can be rearranged as follows and this allows implicit differentiation (posh for chain rule) to be done (as per Stack Exchange [11]).

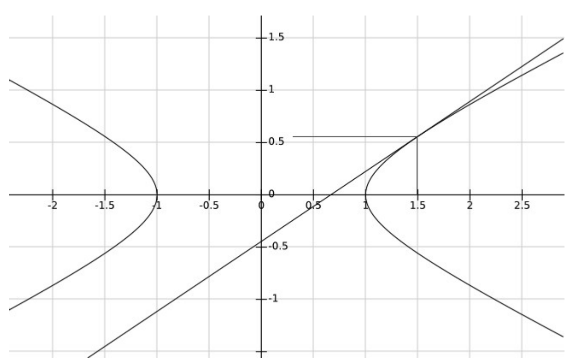
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$
$$\therefore b^2x^2+a^2y^2=a^2b^2$$
$$2b^2x+2a^2y\frac{dy}{dx}=0$$
$$\therefore \frac{dy}{dx}=\frac{-b^2x}{a^2y}$$

The same treatment for the hyperbola gives the same equation, but of opposite sign.

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$
$$\therefore \frac{dy}{dx}=\frac{b^2x}{a^2y}$$

Trying these formulae out for an ellipse and a hyperbola with *a* = 1, *b* = 0.5 gives promising results for the tangent lines at the *x* and *y* values shown below, via fooplot.com:





Subjects still to be covered

The **catenary**, but that'll do for now!

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