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Coulomb Scattering in a Magnetic Field†

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ABSTRACT

It is shown that the contributions of distant encounters to the velocity diffusion coefficients in Coulomb scattering are convergent when a magnetic field is present, and that no cut-off is required in the impact parameter. A cut-off is, however, required in the coefficient which defines space diffusion across the field.

In a plasma the diffusion coefficients are slightly reduced when the typical gyromagnetic radius of the particles is less than the Debye length.

§ 1. INTRODUCTION

IN the Boltzmann equation for an ionized gas it is well known that a magnetic field makes itself felt principally through the Lorentz force term $e(\mathbf{v} \wedge \mathbf{B}) \,.\, \partial f/\partial \mathbf{v}$, in the usual notation. It is not generally appreciated, however, that the collision term $(\partial f/\partial t)_{\mathrm{coll}}$ is also affected, independently, by the field. This is because the field affects the dynamics of ion encounters, angular momentum no longer being conserved. It is shown in the present paper that a magnetic field causes the scattering of ions on ions to fall off faster with increasing impact parameter than in the absence of a field. The velocity diffusion coefficients can therefore be defined without introducing a cut-off in the impact parameter, as is normally required in Coulomb scattering.

§ 2. THE SCATTERING OF AN ELECTRON BY A FIXED PROTON IN A UNIFORM MAGNETIC FIELD

The treatment will be classical and non-relativistic. The equations of motion are then

$$m\frac{d^2\mathbf{r}}{dt^2} = -\frac{e}{c}\frac{d\mathbf{r}}{dt} \wedge \mathbf{B} - \frac{e^2\mathbf{r}}{r^3}, \qquad \ldots \ldots \ldots (1)$$

where $\mathbf{r}$ is the radius vector of the electron relative to the proton, $\mathbf{B}$ is the magnetic field strength, m is the mass of the electron, e is the charge of a proton and c is the velocity of light. Take rectangular axes through the proton, with the z-axis parallel to the field. At large distances from the proton the electron describes a spiral of constant pitch and radius about an axis parallel to the z-axis. Before the encounter suppose that the gyromagnetic axis lies in the y-plane at a distance p from the z-axis. Let the resultant velocity be v, inclined at an angle θ to the z-axis. It is required

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to determine the displacement of the gyromagnetic axis and the change in θ due to the encounter; the resultant velocity, at large distances from the proton, will be unaltered by the encounter since the magnetic field does no work on the electron. An angular momentum integral of (1) also exists, but will not be used in the analysis. On reducing the variables to dimensionless form by substituting

$$t=\mu p\tau/v;\quad \mathbf{r}=p\mathbf{s}=p(\xi,\eta,\zeta),$$

(1) can be written

$$\left.\begin{aligned}\ddot{\varpi}&=i\dot{\varpi}/\lambda-\lambda\mu\varpi/s^3;\\ \ddot{\zeta}&=-\lambda\mu\zeta/s^3,\end{aligned}\right\}\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where $\varpi=\xi+i\eta$, and the differentiations are with respect to τ. λ and μ are given by

$$\lambda=p_1/p;\quad \mu=p_1/p_0;\quad a=mcv/eB;\quad p_0=e^2/mv^2;\quad p_1=(ap_0)^{1/2}.$$

a is the gyromagnetic radius of an electron with velocity v at right angles to the field, and p_0 is the impact parameter for a $90°$ deflection in the absence of the field. p_1 is the distance from the origin at which the Lorentz and Coulomb forces on the electron would have equal strengths. As $p\rightarrow\infty$ with a fixed v and B, $\lambda\rightarrow0$ and μ remains constant.

In the problem of distant encounters, the Coloumb force is weak compared with the Lorentz force, and it can be regarded as a perturbation of the gyromagnetic spiral. The characteristic time of variation of the perturbation is much longer than the gyromagnetic period. The perturbation must therefore be treated by a method similar to the WKB method. ϖ and ζ are expanded in ascending powers of λ, regarding μ as a constant parameter of unit order of magnitude, in the form

$$\left.\begin{aligned}\varpi&=\sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty}\alpha_{mn}(\tau)\lambda^m\exp(in\tau/\lambda);\\ \zeta&=\sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty}\beta_{mn}(\tau)\lambda^m\exp(in\tau/\lambda).\end{aligned}\right\}\quad . \quad . \quad . \quad . \quad (3)$$

ζ is real, hence $\beta_{m,-n}=\overline{\beta_{m,n}}$. The α's and β's vary in a time scale of unit order of magnitude, independently of λ.

Before the encounter the gyromagnetic radius is $\lambda\sin\theta$, in the dimensionless variables, and the gyromagnetic axis is in the η-plane at unit distance from the ζ-axis. Time will be measured from the instant that the electron crosses the ζ-plane. The boundary conditions for the α's and β's are therefore:

$$\left.\begin{aligned}&\alpha_{00}\rightarrow1\\ &\alpha_{11}\rightarrow\sin\theta\exp(i\phi)\\ &\alpha_{mn}\rightarrow0 &&(m,n)\neq(0,0)\text{ or }(1,1)\\ &\dot{\beta}_{00}\rightarrow\mu\cos\theta\\ &\dot{\beta}_{mn}\rightarrow0 &&(m,n)\neq(0,0)\end{aligned}\right\}\text{ as }\tau\rightarrow-\infty,\qquad(4)$$

$$\sum_{n=-\infty}^{\infty}\beta_{mn}=0\qquad\text{all }m\qquad\text{at }\tau=0,$$

where ϕ is an arbitrary parameter determining phase in the spiral orbit. The α's and β's are determined by substituting the expansions (3) into the equations of motion (2) and equating coefficients of $\lambda^m \exp(in\tau/\lambda)$. This produces a set of ordinary linear differential equations of the second order, in which the boundary conditions (4) are sufficient to determine unique solutions. α_{0n}, α_{1n} and β_{0n} for all n can be obtained independently of the higher terms. The coefficients α_{2n}, for all n, are then obtainable in terms of the α_{0n}, etc. All the α's and β's are derivable successively in this way. The expansions (3) are found to be

$$\left.\begin{aligned}\varpi &= 1+\lambda\sin\theta\exp[i(\tau/\lambda+\phi)]-\lambda^2 i\sec\theta(1+hk)+\ldots;\\ \zeta &= h+\lambda\mu^{-1}\sec^2\theta\sinh^{-1}h-\lambda^2\mu^{-2}\sec^4\theta[(2k^{-1}+k)\sinh^{-1}h\\ &\quad +2h\log_e k-\tfrac{7}{2}\tan^{-1}h-2h\log_e 2]+\ldots,\end{aligned}\right\} \qquad (5)$$

where $h=\tau\mu\cos\theta$ and $k=(1+h^2)^{-1/2}$. The higher order terms rapidly increase in complexity.

§ 3. The Displacement of the Gyromagnetic Spiral

The spiral orbit after the encounter is given by

$$\left.\begin{aligned}\varpi_\infty &= \sum_{m=0}^{\infty}\alpha_{m0}(\infty)\lambda^m+\exp(i\tau/\lambda)\sum_{m=0}^{\infty}\alpha_{m1}(\infty)\lambda^m;\\ \dot{\zeta}_\infty &= \sum_{m=0}^{\infty}\dot{\beta}_{m0}(\infty)\lambda^m.\end{aligned}\right\} \qquad (6)$$

The gyromagnetic axis is therefore displaced, due to the encounter, by an amount given by

$$\Delta\varpi = \sum_{m=0}^{\infty}\Delta\alpha_{m0}\lambda^m, \qquad (7)$$

where $\Delta\alpha_{m0}=\alpha_{m0}(\infty)-\alpha_{m0}(-\infty)$. The change in velocity along the field is given by

$$\Delta\dot{\zeta} = \sum_{m=0}^{\infty}\Delta\dot{\beta}_{m0}\lambda^m, \qquad (8)$$

where $\Delta\dot{\beta}_{m0}=\dot{\beta}_{m0}(\infty)-\dot{\beta}_{m0}(-\infty)$. From the expansions (5)

$$\Delta\varpi = -2\lambda^2 i\sec\theta+0(\lambda^3); \qquad (9)$$

$$\Delta\dot{\zeta} = 12\lambda^6\mu^2\sin\theta+0(\lambda^8). \qquad (10)$$

§ 4. Convergence of the Diffusion Coefficients

Consider an electron moving among fixed protons at a density n per cm^3, in the presence of a uniform magnetic field. Suppose that the Debye length in the plasma is large compared with the gyromagnetic radius of the electron, so that the motion of the electron is a series of encounters, between which steady gyromagnetic spirals are described. The statistical behaviour of the electron is then describable by three quantities, namely the diffusion of the gyromagnetic axis in space, the mean rate of decrease of velocity along the field, and the diffusion, in velocity space, of the velocity perpendicular to the field.

Dealing first with the gyromagnetic axis, by symmetry there will be no mean drift, but there will be a statistical spread radially outwards from an initial position. The mean number of encounters per second in which the encounter parameters lie within the ranges $(\phi, \phi+d\phi)$, $(\theta, \theta+d\theta)$, $(v, v+dv)$ and $(p, p+dp)$ is $n \sin\theta \cos\theta\, vf(v)p\, dp\, dv\, d\theta\, d\phi$, where f is the probability of an electron having a resultant velocity within the range $(v, v+dv)$. Suppose that the gyromagnetic axis is at some specified point at a given instant. Then the mean square displacement from this position, after a time dt is given by

$$\langle |\Delta w|^2 \rangle\, dt = n\, dt \int_{v=0}^{\infty} \int_{p=0}^{p_m} \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} |\Delta w|^2 vfp \sin\theta \cos\theta\, d\phi\, d\theta\, dp\, dv, \quad (11)$$

where p_m is the maximum value of the impact parameter beyond which encounters cannot be taken as binary. It has been shown (Spitzer 1956) that in a plasma p_m is of the order of magnitude of the Debye length. $\langle |\Delta w|^2 \rangle$ is the rate of increase of the mean square displacement. In distant encounters Δw, as determined by (9), is given by

$$w = -2iap_0 p^{-1} \sec\theta, \quad (12)$$

provided that θ is not arbitrarily near $\pi/2$. The expression on the right-hand side of (11) therefore increases indefinitely as p_m increases. The diffusion coefficient therefore cannot be defined without introducing this cut-off parameter.

The mean rate of change of velocity parallel to the field, i.e. the dynamical friction, is given by

$$\langle \Delta v_{\parallel} \rangle = n \int_{v=0}^{\infty} \int_{p=0}^{p_m} \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} \Delta\dot{z} vfp \sin\theta \cos\theta\, d\phi\, d\theta\, dp\, dv.$$

In considering the contribution from distant encounters $\Delta\dot{z}$ can be obtained from (10), thus

$$\Delta\dot{z} = 12a^{7/2} v p_0^{5/2} p^{-6} \sin\theta. \quad (13)$$

$\langle \Delta v_{\parallel} \rangle$ therefore tends to a finite limit as $p_m \rightarrow \infty$. The coefficient of dynamical friction can therefore be defined without introducing the Debye cut-off parameter, thus

$$\langle \Delta v_{\parallel} \rangle = n \int_{v=0}^{\infty} \int_{p=0}^{\infty} \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} \Delta\dot{z} vfp \sin\theta \cos\theta\, d\phi\, d\theta\, dp\, dv. \quad (14)$$

The mean square diffusion coefficients $\langle (\Delta v_{\parallel})^2 \rangle$ and $\langle (\Delta v_{\perp})^2 \rangle$, in velocity space, parallel and perpendicular to the field, respectively, can be similarly defined without a cut-off parameter.

§ 5. Conclusions

If the magnetic field in a plasma is so strong that the typical gyromagnetic radius of the particles is small compared with the Debye length then the particles, between collisions, describe well-defined gyromagnetic spirals. The statistical behaviour of the plasma can then be described in terms of the diffusion of the axes of the gyromagnetic spirals, due to encounters, together with the usual velocity diffusion coefficients $\langle \Delta v_{\parallel} \rangle$, $\langle (\Delta v_{\parallel})^2 \rangle$ and $\langle (\Delta v_{\perp})^2 \rangle$. The space diffusion coefficient for the axes is not convergent for large values of the impact parameter in encounters, and it requires a Debye cut-off. The velocity diffusion coefficients, however, are convergent and can be defined without a cut-off. They are, moreover, reduced by the magnetic field, as compared with the values computed for pure Coulomb scattering. The transport phenomena, derived with the use of Boltzmann's equation, will therefore be modified. The Debye length in a plasma is $(kT/4\pi n_e e^2)^{1/2}$, where k is Boltzmann's constant, n_e is the electron density and T is the temperature. The gyromagnetic radius for an electron at the mean thermal velocity

$$\bar{v} = (3kT/m)^{1/2} \quad \text{is} \quad (mc/eB)(3kT/m)^{1/2};$$

it is therefore less than the Debye length if $B > c(12\pi m n_e)^{1/2}$. In the solar chromosphere, at a level where $n_e = 10^{11}$, the condition is $B > 2 \times 10^3$ gauss. This is the typical field strength in a sunspot. At a density $n_e = 10^{14}$, $B > 6 \times 10^4$ gauss. Such a situation could arise in laboratory discharges. The modification to the diffusion coefficients, under these conditions, would only be slight. Close electron–proton encounters, namely those where $p \sim p_0$, are seriously affected by the field only if $p_1 \sim p_0$, i.e. if $a \sim p_0$. Taking v to be the mean thermal velocity of the electrons this requires $B \sim m^{1/2} c (3kT)^{3/2}/e^3 \simeq 100 T^{3/2}$. Under astrophysical and laboratory conditions this would involve impossibly high field strengths.

Reference

Spitzer, L., 1956, *Physics of Fully Ionized Gases*, Ch. 5 (New York : Interscience Publishers).