

Electrical and optical properties of materials

John JL Morton

Part 4: Maxwell's Equations

We have already used Maxwell's equations for electromagnetism, and in many ways they are simply a reformulation (or even just a copy) of equations you have already come across and which were established before Maxwell. However, the assimilation of a set of existing knowledge in a particular way can be extremely valuable. Furthermore, Maxwell derived three of these equations independently, and in doing so made a critical correction which makes possible the electromagnetic wave. We shall see that Maxwell's equations will unify electricity, magnetism and even light. In this regard, it is entirely appropriate that we know the set as 'his' equations.

4.1 A reminder on waves

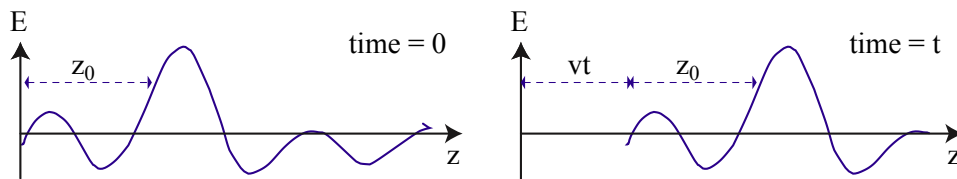


Figure 4.1: Travelling waves

A travelling wave has some arbitrary wave-form which remains fixed in shape and propagates through space at some velocity v . A 'real-world' wave has some amplitude representing some physical property (e.g. electric field E) which becomes a function of both distance and time. Suppose the wave propagates along the z axis (see Figure 4.1). At time $t = 0$ the amplitude of E has some form $E = f(z)$. At time t the waveform has advanced a distance vt along z . The equation now representing the same wave has shifted:

$$E = f(z - vt) \quad (4.1)$$

This general equation describes any *non-decaying* wave. Defining $u = z - vt$, we use the product rule to differentiate E with respect to z :

$$\frac{\partial E}{\partial z} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial z}, \quad \text{and} \quad \frac{\partial u}{\partial z} = 1 \quad (4.2)$$

Repeating this differentiation, with respect to z ,

$$\frac{\partial^2 E}{\partial z^2} = \frac{\partial^2 f}{\partial u^2} \frac{\partial u}{\partial z} = \frac{\partial^2 f}{\partial u^2} \quad (4.3)$$

4. Maxwell's equations for electromagnetism

Then we do the same with respect to time:

$$\frac{\partial E}{\partial t} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial t}, \quad \text{here} \quad \frac{\partial u}{\partial t} = -v \quad (4.4)$$

Repeating this differentiation, with respect to z ,

$$\frac{\partial^2 E}{\partial t^2} = -v \frac{\partial^2 f}{\partial u^2} \frac{\partial u}{\partial t} = v^2 \frac{\partial^2 f}{\partial u^2} \quad (4.5)$$

Combing Eq. 4.3 and 4.5 above gives us the wave equation:

$$\frac{\partial^2 E}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 E}{\partial t^2} \quad (4.6)$$

Waves with sine or cosine amplitude are solutions to the above equation (reassuringly!). For example, take this cosine wave with wavelength λ :

$$E = E_0 \cos\left(\frac{2\pi}{\lambda}(z - vt)\right) \quad (4.7)$$

Defining the *wavenumber* $k = 2\pi/\lambda$, and remembering that the velocity $v = f\lambda$ where f is the frequency:

$$E = E_0 \cos(kz - 2\pi ft) \quad (4.8)$$

We often use the *angular frequency* $\omega = 2\pi f$ (it has units of radians per second).

$$E = E_0 \cos(kz - \omega t) \quad (4.9)$$

Plugging this into the wave equation Eq. 4.6 gives:

$$k^2 E = \frac{1}{v^2} \omega^2 E \quad \text{or} \quad v = \omega/k \quad (4.10)$$

Finally, remember that we can use $\exp(i\phi) = \cos(\phi) + i\sin(\phi)$ as a combination of cosine and sine solutions for a general wave. Thus, defining a general wave as $E = E_0 \exp[i(kx - \omega t)]$ is quite common.

4.2 Defining Maxwell's equations

4.2.1 Maxwell 1 and 2

The first two Maxwell's equations are really a formulation of Gauss's Theorem for electric and magnetic 'charges', and are best appreciated using the picture of vector fields. We are used to drawing vector fields to describe

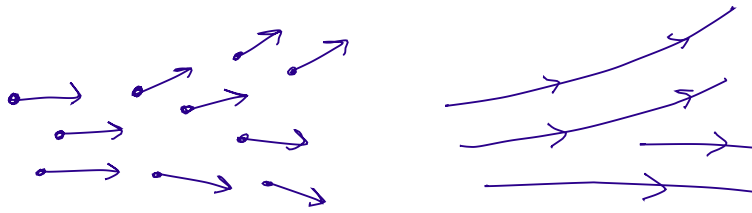


Figure 4.2: A vector field

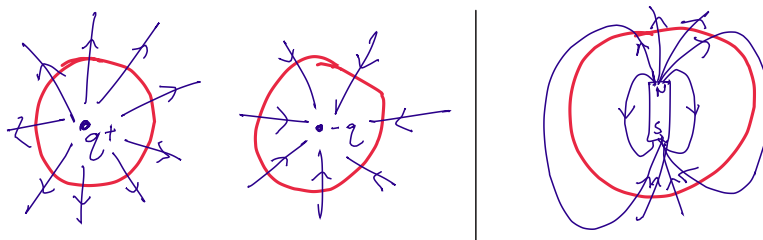


Figure 4.3: Left and centre: a positive or negative electric charge will act as a sink or source of electric field lines. The total electric field flux through the surface (red) is equal to the charge contained within (Maxwell **1**). The same applies for magnetic fields, but as we always have magnetic dipoles, all surfaces will have net magnetic field flux of 0 (Maxwell **2**).

how a field is oriented and varies through space (e.g. wind), as shown in Figure 4.2. Maxwell's first equation states that the net flux of electric field \mathbf{E} through any closed surface is equal to the net charge contained inside, divided by the permittivity ϵ . This is illustrated in Figure 4.3 for positive and negative charges, which act as sinks or sources for an electric field. However, if we try to make the same picture for a magnetic dipole, we see that the net flux through any surface is always zero (precisely because it is a *dipole*, not a *monopole*). No magnetic monopoles or magnetic 'charges' exist — or more accurately *have yet been found*. Hence, Maxwell's second equation is as above, but for magnetic fields where the magnetic 'charge' contained is always 0.

These statements can be formalised mathematically by taking the sum of the derivative of each component of the field along each direction x , y and z .

4. Maxwell's equations for electromagnetism

Maxwell's Equations

- 1** The flux of \mathbf{E} through any closed surface is equal to the net charge contained inside $/\epsilon$

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \rho/\epsilon$$

- 2** The flux of \mathbf{B} through any closed surface is zero

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0$$

4.2.2 Maxwell 3 and 4

We live in a three dimensional world (well arguably more...), and so it is clear that any general theory of electromagnetism must handle electric and magnetic fields which point and propagate in any direction. Maxwell's next two equations in their general form do this, however this requires a mathematical toolbox (namely the vector operators div, grad and curl) which lie beyond the scope of this course. However, we can probe the richness of these equations and their conclusions in a set of restricted dimensions. Let us state that the electric and magnetic fields are only allowed variation in time and along a direction z (as would be the case for some plane-wave propagation in one direction).

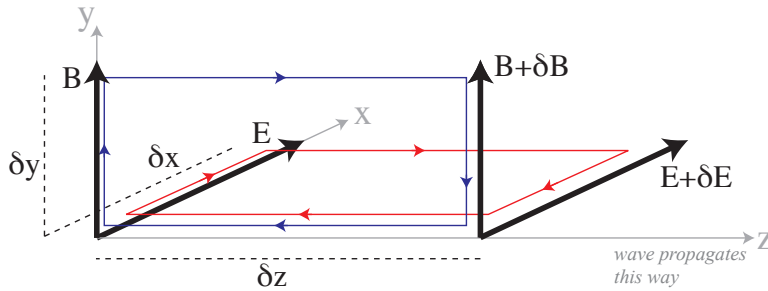
Maxwell's Equations in 1D (ish)

3
$$\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t}$$

4
$$\frac{\partial B_y}{\partial z} = -\mu \left(J_x + \epsilon \frac{\partial E_x}{\partial t} \right)$$

We can derive the next two of Maxwell's equations for this case of propagation in one dimension, with the help of Figure 4.4. We have defined the electric field E to point along x , and the magnetic field B to point along y (Note¹). We can begin by using Faraday's Law of induction around the loop in red ($x - z$). Faraday's Law states that the voltage around this loop is

¹This doesn't necessarily take away from the generality of the result — we are working out the behaviour of the magnetic field orthogonal to the electric field, which in general could point anywhere in the $x - y$ plane.

Figure 4.4: Deriving Maxwell's equations **3** and **4** for propagation in 1D

equal to the rate of change of flux through it:

$$(E\delta x - (E + \delta E)\delta x) = \frac{\delta B_y}{\delta t} \delta x \delta z \quad (4.11)$$

Rearranging gives:

$$\frac{\delta E_x}{\delta z} = -\frac{\delta B_y}{\delta t} \quad (4.12)$$

Now we'll use Ampère's Law around the blue loop ($y-z$): this states that the integral of magnetic field around a loop is equal to the current running through the loop, times μ , the permeability.

$$(B\delta y - (B + \delta B)\delta y) = \mu J_x \delta y \delta z \quad (4.13)$$

$$\frac{\delta B_y}{\delta z} = -\mu J_x \quad (4.14)$$

We have multiplied the current density J by the area of the loop to get a current. Now, there is an all-important correction to Ampère's Law which Maxwell introduced into this equation. He postulated that there was an additional current density term $= \epsilon \partial E_x / \partial t$ which we now call the *displacement current*^[2].

$$\frac{\partial B_y}{\partial z} = -\mu \left(J_x + \epsilon \frac{\partial E_x}{\partial t} \right) \quad (4.15)$$

There was no *a priori* reason why it should be there, but in adding it Maxwell was able to define the electromagnetic wave, and the velocity with which it may travel. Experiments conducted years later by Hertz were able to confirm Maxwell's insight and the significance of the relations he had derived.

²You may also see Maxwell's equations in terms of $D = \epsilon E$ and $H = B/\mu$. The displacement current is then $\partial D / \partial t$

4.3 Electromagnetic waves in a vacuum

We'll begin by taking a look at Maxwell's Equations **3** and **4** in a vacuum, where the current $J = 0$. We can take **3** and differentiate both sides with respect to z :

$$\frac{\partial^2 E_x}{\partial z^2} = -\frac{\partial}{\partial z} \frac{\partial B_y}{\partial t} \quad (4.16)$$

$$\frac{\partial^2 E_x}{\partial z^2} = -\frac{\partial}{\partial t} \frac{\partial B_y}{\partial z} \quad (4.17)$$

Substitute in **4** on the right hand side:

$$\frac{\partial^2 E_x}{\partial z^2} = \frac{\partial}{\partial t} \left(\mu_0 \epsilon_0 \frac{\partial E_x}{\partial t} \right) \quad (4.18)$$

$$\frac{\partial^2 E_x}{\partial z^2} = \mu_0 \epsilon_0 \frac{\partial^2 E_x}{\partial t^2} \quad (4.19)$$

Looking back at Eq. 4.6 we see that this is nothing else than the wave equation. Thus, electric and magnetic fields propagate as waves with velocity:

$$\frac{1}{v^2} = \mu_0 \epsilon_0 \quad \text{or} \quad v = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \quad (4.20)$$

The values of μ_0 and ϵ_0 were known at the time of Maxwell's work from measurements by Ampère, Faraday and others:

$$\mu_0 = 4\pi \times 10^{-7} \text{ Hm}^{-1} \quad \text{and} \quad \epsilon_0 = 8.854 \times 10^{-12} \text{ s}^2\text{H}^{-1}\text{m}^{-1} \quad (4.21)$$

Thus, Maxwell was able to calculate the speed with which electromagnetic waves travel through a vacuum (or 'free space'):

$$c_0 = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 299\,792\,458 \text{ ms}^{-1} \quad (4.22)$$

4.4 Electromagnetic waves in non-conducting, uncharged dielectrics

The derivation above assumed the simplest case of a vacuum. We can generalise this somewhat to a non-conducting (J remains 0) material with relative permittivity ϵ_r and permeability μ_r . The same steps give:

$$c = \frac{1}{\sqrt{\mu_0 \mu_r \epsilon_0 \epsilon_r}} = \frac{c_0}{\sqrt{\epsilon_r \mu_r}} \quad (4.23)$$

As ϵ_r and μ_r are always greater than 1, this means light never travels faster than in a vacuum. The *refractive index* n of a material describes by what factor light slows down in materials:

$$n = \frac{c_0}{c} = \sqrt{\epsilon_r \mu_r} \quad (4.24)$$

4.5 Impedance

The *impedance* Z is defined as the ratio of the strengths of the electric field E_x and magnetizing field H_y ($= B_y/\mu$) [Note³]. Now that we know they are waves, we can call the electric field $E_x = E_0 \exp(i(kz - \omega t))$, and the magnetic field $B_y = B_0 \exp(i(kz - \omega t))$. From **Maxwell 3** we have:

$$ikE_0 \exp(i(kz - \omega t)) = i\omega B_0 \exp(i(kz - \omega t)) \quad (4.25)$$

$$\frac{E_0}{B_0} = \frac{\omega}{k} = c \quad (4.26)$$

The impedance is therefore:

$$Z = \frac{E_x}{H_y} = \frac{\mu E_0}{B_0} = \mu c = \frac{\mu}{\sqrt{\mu\epsilon}} = \sqrt{\frac{\mu}{\epsilon}} \quad (4.27)$$

The impedance of free space is:

$$Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 377 \, \Omega \quad (= 120\pi \, \Omega \text{ is a common approx.}) \quad (4.28)$$

Similarly, in an isotropic, charge-less material, the *characteristic impedance* is:

$$Z = \mu_r \mu_0 c = \sqrt{\frac{\mu_0 \mu_r}{\epsilon_0 \epsilon_r}} \quad (4.29)$$

4.6 Electromagnetic waves in conducting dielectrics

For a conducting dielectric we can no longer assume the current density is zero. We must therefore go through the derivation in Eqs 4.16 through 4.19 again, without neglecting J :

$$\frac{\partial^2 E_x}{\partial z^2} = -\frac{\partial}{\partial z} \frac{\partial B_y}{\partial t} \quad (4.30)$$

$$\frac{\partial^2 E_x}{\partial z^2} = -\frac{\partial}{\partial t} \frac{\partial B_y}{\partial z} \quad (4.31)$$

Substitute in **Maxwell 4** on the right hand side:

$$\frac{\partial^2 E_x}{\partial z^2} = \mu \frac{\partial}{\partial t} \left(J_x + \epsilon \frac{\partial E_x}{\partial t} \right) \quad (4.32)$$

³Why do we use the magnetizing field H and not the magnetic field B ? The units of H are amps per metre and those of E are volts per metre. The ratio of E and H therefore has units of volts per amp (or ohms) the units for an impedance

4. Maxwell's equations for electromagnetism

In a conducting medium, the current density J is related to the applied electric field E by the relation $J = \sigma E$:

$$\frac{\partial^2 E_x}{\partial z^2} = \mu \frac{\partial}{\partial t} \left(\sigma E_x + \epsilon \frac{\partial E_x}{\partial t} \right) \quad (4.33)$$

$$\frac{\partial^2 E_x}{\partial z^2} = \mu \sigma \frac{\partial E_x}{\partial t} + \mu \epsilon \frac{\partial^2 E_x}{\partial t^2} \quad (4.34)$$

This resulting equation is similar to the wave equation. Let's therefore try a trial solution with the same form of wave: $E = E_0 \exp[i(kz - \omega t)]$. Plugging it in, we see that it is indeed a solution, with:

$$k = \sqrt{\mu \epsilon \omega^2 + i \omega \mu \sigma} \quad (4.35)$$

For $\sigma = 0$ we obtain the same results as derived above for non-conducting media. However, we see that for non-zero σ , the wavenumber k acquires some imaginary component. This imaginary component results in a *damping* of the wave with distance. For complex k ,

$$E = E_0 e^{i(kz - \omega t)} = E_0 e^{i(\text{Re}[k]z - \omega t)} e^{-\text{Im}[k]z} \quad (4.36)$$

A very reasonable assumption for a metal is that the conduction current (which goes as σ) is much greater than the displacement current (which goes as $\epsilon \omega$). Thus, for a good conductor we can say that $\sigma \gg \epsilon \omega$ and the above relation for δ simplifies to⁴:

$$\text{Re}[k] = \text{Im}[k] = \sqrt{\frac{\sigma \mu \omega}{2}} \quad (4.37)$$

The decay constant of the damping with distance is thus $1/\text{Im}[k]$. We call this decay constant the *skin depth* δ :

$$\delta = \sqrt{\frac{2}{\sigma \mu \omega}} \quad (4.38)$$

Some examples: Copper has $\sigma_{Cu} = 10^8 \text{ Sm}^{-1}$, $\mu_r = 1$, so for 1 MHz the skin depth is only $\delta = 65 \text{ } \mu\text{m}$. At 10 GHz it falls to 650 nm. Materials with small skin depths therefore become ineffective at transmitting electromagnetic waves at high frequencies and the wave only travels along the outer 'skin' of the conductor and the effective resistance increases.

High voltage overhead transmission lines often use aluminium cable (with a steel reinforcing core) for this reason. The steel core has high-resistivity and can be neglected — due to the skin effect the current is carried primarily by the outer aluminium portion of the cable δ_{Al} @ 50 Hz = 11 mm.

⁴Reminder: $\sqrt{i} = (1 + i)/\sqrt{2}$

4.7 Energy flow and the Poynting vector

The energy stored (per unit volume) in electromagnetic fields is:

$$U = \frac{1}{2}(\mathbf{D} \cdot \mathbf{E} + \mathbf{B} \cdot \mathbf{H}) = \frac{1}{2}(\epsilon E_x^2 + \mu H_y^2), \quad (4.39)$$

in the case of our wave propagating in the z direction. Give a wave velocity c , the energy flux (per unit area) is:

$$cU = \frac{1}{2} \left(\sqrt{\frac{\epsilon}{\mu}} E_x^2 + \sqrt{\frac{\mu}{\epsilon}} H_y^2 \right) = \frac{1}{2} \left(\frac{E_x^2}{Z} + Z H_y^2 \right), \quad (4.40)$$

where Z is the impedance derived above. We defined the impedance to be the ratio of E_x to H_y , so

$$flux = cU = E_x H_y = \frac{E_x^2}{Z} = Z H_y^2 \quad (4.41)$$

We see the energy is stored equally in the electric and magnetic fields. The *Poynting vector* gives a measure of the direction and flux of energy flow in electromagnetic fields:

$$\mathbf{P} = \mathbf{E} \times \mathbf{H} \quad (P_z = E_x H_y) \quad (4.42)$$