1 The Paper

Philosophy of mathematics lies at the deep end of epistemology and metaphysics. Its problems have absorbed the most powerful minds from Plato on. Those problems form the subject of this paper. You do not need to know a great deal of mathematics; if you have been exposed to basic school mathematics plus a touch of calculus you will be able to feel the full force of the philosophical mysteries. However, some knowledge of mathematical logic brings parts of the subject into much sharper focus; thus the paper forms a natural pair with the Symbolic Logic paper, but neither paper depends on the other. Although the paper is not divided into sections, the subject matter can be looked in two ways, thematically and historically.

a. The subject matter

Thematically, the subject centres on the following problem: how can we provide accounts of the nature of (a) mathematical reality and (b) mathematical knowledge, which are internally plausible and mutually compatible?

Mathematical reality is in itself mysterious: how can it be highly abstract and yet applicable to the physical world? How can mathematical theorems be necessary truths about an unchanging realm of abstract entities and at the same time so useful in dealing with the contingent, variable and inexact happenings evident to the senses?

Mathematical knowledge is no less puzzling. True, we know some things by proving them; but if a proof is to deliver knowledge of its conclusion, its premisses must be known; on pain of an infinite regress there must be some truths known without proof: axioms. How do we know the axioms? And what exactly is a proof anyway? There is ample evidence that people can come to see some mathematical truths for themselves, without deducing them from things they have been told. What modes of cognition are involved? What makes it reasonable to trust them? All these problems are intensified once we move from finite mathematics to the infinite.

Finally, there is a special problem for any comprehensive philosophy of mathematics: how can a plausible account of mathematical reality be coherently combined with a plausible account of mathematical knowledge? How can we finite physical creatures have knowledge of the infinite non-physical structures of mathematics?

Historically, the starting point is Plato who proposed that mathematical reality consists of perfect forms independent of the physical world. This view of the subject matter of mathematics lies at one end of a spectrum of metaphysical views; towards the other end is the view is that the subject matter is a purely human artefact. Views towards the Platonic end are known as Platonist; towards the other end, anti-Platonist. That is a classification of metaphysical views. Epistemological views fall into two classes, roughly speaking mathematical truths are known (i) by reason, or (ii) by inference from the evidence of the senses supplemented by deduction. There are a few important epistemological views which fall into neither camp, notably those of Plato, Kant and Gödel.

In the 16th-17th century there was some epistemological agreement across the rationalist/empiricist divide: Descartes and Leibniz, both great mathematicians, thought that mathematical truths are known by reason rather than by inference from the evidence of the senses; Locke and Hume agreed. But they disagreed famously about the origin of our mathematical concepts, the rationalists (following Plato) holding that they are in some sense innate, the empiricists holding that they are in some way derived from sense experience. This debate has recently been revived by work of cognitive psychologists which suggests that the basis of our grasp of concepts of e.g. cardinals 1, 2, and 3 has a significant innate component.

In the 18th Century Kant boldly proposed that the structure of our minds provides the basis of both mathematical reality and our knowledge of it. This was opposed in the 19th Century by Mill; he thought that arithmetical and geometrical truths are generalisations about the natural world known by induction from sense experience or by deduction from such generalisations. Developments in 19th century mathematics now burst in: the (non-empirical) discovery of non-Euclidean geometries refuted not only views of Kant and Mill, but also undermined the time-honoured view that geometry is the paradigm of justified certainty; no less important are spin-offs of the attempt to give calculus a rigorous formulation, e.g. the discovery of functions in the plane which have no curve, and Cantor's development of a theory of infinite numbers.

The basis of the drive for rigour was Cantor's theory of classes; this was used by Dedekind and Frege (in quite different ways) in an attempt to provide 'logical' foundations for non-negative integer arithmetic and thence for all mathematics. The view that logic encompasses mathematics and provides its epistemic basis is known as 'logicism', a version of the idea that mathematics is known by reason rather than sense experience. At the end of the 19th century Cantor discovered certain paradoxes within the theory of classes, and soon after Russell discovered that Frege's precise version of the theory is inconsistent. This produced a crisis: philosophy of mathematics in the early decades of the 20th century was a response to the class paradoxes.

Russell's theory of types, a version of logicism, and Hilbert's formalist philosophy proposed different ways of providing a consistent foundation for mathematics. Meanwhile Brouwer, under the banner of intuitionism, urged that mathematics has no foundations. In an epoch-making paper Gödel quietly proved that mathematics does not have foundations of the kinds envisaged by Russell and Hilbert. This and subsequent findings in mathematical logic have led to the demise of the view that all of mathematics has some indubitable foundation.

In its place two main schools came to the fore: constructivism, according to which only a restricted part of mathematics, that which we can in some sense construct, can be justifiably regarded as true; Quine's empiricist view, according to which we can justifiably regard as true that part of mathematics which is indispensable to natural science. Constructivists are invariably anti-Platonists; Quinian empiricists tend to be Platonists.

Constructivism takes various forms, chiefly predicativism (Poincaré, Russell, Feferman) and intuitionism (Brouwer, Heyting, Dummett). Explanation of these isms is impossible here. Quine's view has recently come under fire, but has also been defended and extended (Resnik); and some new empiricist views have emerged (Kitcher, Maddy). There remains also Gödel's view that we have some non-sensory mode of cognising elements of mathematical reality, which is in no way a human construction. Recent efforts to avoid commitment to abstract independent objects has led to a variety of views; mathematics is false but useful (Field, Papineau), mathematics is about possible constructions (Chihara); mathematics is about structural possibilities (Hellman). These are anti-Platonist views. A view favoured by Platonists who want to avoid the idea that we have some non-sensory mode of apprehending mathematical objects is that pure mathematics is about abstract structures (Resnik, Shapiro).

At present there is a dearth of fine-grained work in epistemology of mathematics. Exceptions are studies of the epistemology of the infinite (Lavine) and of some simple arithmetical and geometrical beliefs (Giaquinto). Recent empirical work on numerical cognition is likely to be valuable (Butterworth), given that the ways in which we actually acquire numerical beliefs might suffice to deliver knowledge.

b. Approaching the subject

This account of the subject matter is just a sketch. But it should be clear that one could not cover it all in two years. The purpose of the sketch is to help you locate the four or five topics you choose to concentrate on. In choosing your topics you must take into account (a) which topics most catch your imagination as a result of your reading and the lectures, and (b) which topics are most regularly and most straightfowardly covered in the exam paper. In setting the exam the examiners will ask for questions from those who have lectured on the subject over the past couple of years, so it is wise to attend some lecture courses.

To make best use of the lecture courses you should try to get clear about the topic under discussion: What precisely are the positions? What precisely are the arguments for and against? There is no substitute for formulating your answers in writing between lectures, so that by the end of the lecture course you have a substantial body of written material to draw on when preparing for exams and writing pre-submissions. What you get out of lectures can be re-inforced and deepened by reading.

***

This bibliography is selective in two ways. First, many articles, including good ones, on topics dealt with here are not mentioned. Secondly, whole topics are not dealt with e.g. the nature of proof, and the metaphysical nature of the transfinite. Writings on the philosophical significance of results in mathematical logic have also been omitted, apart from Gödel's incompleteness theorems. This does not adequately explain why certain notable contributors to the subject are here under-represented, for example, Lakatos, Putnam, Parsons (C), and others. But one has to stop somewhere.

2 Basic Reading

You will have to read some of the primary sources e.g. the exchange between Socrates and the slave boy in Plato's *Meno*. Sometimes secondary reading can give a helpful preview of primary material e.g. Furth's introduction to Frege's *Basic Laws of Arithmetic*. But there is a huge literature and a consequent retrieval problem. Here annotated bibliographies are useful. Lecturers will supply bibliographies of relevant material: if you get a long list of items with no indication of relative importance or relative difficulty, do not hesitate to ask the lecturer where to start and how to go on.

The most useful collection of readings for this paper is

Benacerraf, Paul, and Hilary Putnam, eds. 1983. *Philosophy of Mathematics: Selected Readings.* 2nd ed. Cambridge: Cambridge University Press. Referred to as B&P below.

A useful recent collection is

Hart, W. D. ed. 1996. *The Philosophy of Mathematics.* Oxford: Oxford University Press.

To bolster your acquaintance with mathematics itself

Allen, R. G. D. 1962. *Basic Mathematics.* London: MacMillan.

Waismann, F. 1951. *Introduction to Mathematical Thinking: the Formation of Concepts in Modern Mathematics.* London: Hafner Publishing Co.

The introduction to Benacerraf & Putnam (op. cit.) gives a helpful overview.

See also

Parsons, C. 1967. 'Foundations of Mathematics'. In P. Edwards, ed., *The Encyclopaedia of Philosophy.* London: Collier-Macmillan.

There are several general introductions to philosophy of mathematics. Avoid them: they tend to be unreliable and your time is better spent on basic material.

3 Historical

Here is a ruthlessly selective list. Nothing marginal, technical or inaccessible is included.

Plato

Meno, §§82b9-85b7.

Phaedo, §§72e-77d.

Republic, §§507a-511e, 525d-527c.

Wedberg, A. 1955. *Plato's Philosophy of Mathematics.* Stockholm: Almquist & Wiksell.

Giaquinto, M. 1993. 'Diagrams: Socrates and Meno's Slave'. *International Journal of Philosophical Studies* 1: 81-97.

Aristotle

Metaphysics M3, Physics B2. For more precise references see Lear below.

Lear, J. 1982. 'Aristotle's Philosophy of Mathematics'. *Philosophical Review* 91: 161-192.

Stich, Stephen P. 1975. ed. *Innate Ideas.* London: University of California Press. Look at this for the 17th century debate see the writings by Descartes, Locke and Leibniz.

Berkeley

His writing on Philosophy of Mathematics is scattered. See the references in the very useful study:

Jesseph, D. 1993. *Berkeley's Philosophy of Mathematics.* Chicago: University of Chicago Press.

Kant

The Critique of Pure Reason, translated by N. Kemp Smith, 2nd ed., London: Macmillan, 1933. Or translated by Paul Guyer, and Allen W. Wood, Cambridge: Cambridge University Press, 1997. Introduction & Transcendental Aesthetic.

Prolegomena to any Future Metaphysics, trans. L. W. Beck, Indianapolis: Bobbs-Merrill, 1950. Or translated by Paul Carus, rev. by James W. Ellington, Indianapolis: Hackett Pub. Co., 1977. First Part of the Main Transcendental Question, How is Mathematics Possible?

Walker, Ralph C. S. 1982. ed., *Kant on Pure Reason*. Oxford: Oxford University Press. Papers by Parsons, Hopkins. See also §§4a, 4b of its bibliography.

Parsons, C. 1992. 'The Transcendental Aesthetic'. In *The Cambridge Companion to Kant,* edited by Paul Guyer. Cambridge: Cambridge University Press.

Friedman, M. 1992. *Kant & the Exact Sciences*. Cambridge, Mass.: Harvard University Press. Chs.1 & 2.

J.S. Mill

A System of Logic. London: Longman, 1843. Bk.II, Chs. 5, 6, 7.

Kim, J. 1981. 'The role of Perception in *A Priori* Knowledge'. *Philosophical Studies* 40: 339-354.

An important collection of primary material is:

Ewald, W. B. ed. 1996. *From Kant to Hilbert: Readings in the Foundations of Mathematics.* 2 Vols. Oxford: Clarendon. Vol. I, II, 1996.

4 Selective Reading for the Modern Period

The discovery of the paradoxes (1895-1905) opens a new period. The writing is inevitably more technical, more involved with issues of philosophical logic; but the fundamental epistemological and metaphysical problems have not changed. At the end of an entry [i] signifies an introductory non-technical item, [d] signifies a difficult item and [t] a technical item.

Fraenkel, A. A., J. Bar-Hillel, and A. Levy. 1973. *The Foundations of Set Theory*. 2nd rev. ed. Amsterdam: North-Holland Publishing Co. Ch. 1, 'The Antinomies', gives a full statement of all the paradoxes. [t]

Maniosu, P. ed. 1998. *From Brouwer to Hilbert: The Debate on the Foundations of Mathematics in the 1920's.* New York: Oxford University Press. A collection of articles by major contributors to the debate in the 20's with helpful editorial introductions. [d]

Feferman, S. 1998. *In the Light of Logic.* Oxford: Oxford University Press. Ch. 2, 'Infinity in mathematics: Is Cantor necessary?'. A brilliant overview of the period from Cantor to Gödel by a top logician. [t]

A. LOGICISM

i. Frege

Frege, G. 1884. *Foundations of Arithmetic: a Logico-mathematical Enquiry into the Concept of Number.* Translated by J. L. Austin. 2nd ed. Oxford: Blackwell, 1953.

For deeper investigation of Frege's logicism, see Frege's introduction to

G. Frege, *The Basic Laws of Arithmetic: Exposition of the System.* Translated by Monygomery Furth. Berkeley: University of California Press, 1964. Furth's substantial preface to the above is especially helpful. For Frege's response to the inconsistency found in his system by Russell, see Appendix II. [t]

Resnik, M. 1980. *Frege and the Philosophy of Mathematics.* Ithaca: Cornell University Press.

Dummett, M. 1991. *Frege: Philosophy of Mathematics*. London: Duckworth.

For recent assessments of Frege's contribution see the papers collected in

Demopoulos, W. ed. 1995. *Frege's Philosophy of Mathematics.* Cambridge, Mass.: Harvard University Press.

ii. Russell

The philosophy underlying *Principia Mathematica* is logicism constrained by predicativism. This view is explained in

Russell, B. 1908. 'Mathematical Logic as Based on the Theory of Types'. In *Logic and Knowledge: Essays 1901-1950*, ed. R. C. Marsh. London: Allen & Unwin, 1956; also in *Frege and Gödel: two Fundamental Texts in Mathematical Logic*, edited by Jan van Heijenoort. Cambridge, Mass.: Harvard University Press, 1970. ([t][d], but not philosophically marginal.)

Russell, B. 1919. *Introduction to Mathematical Philosophy*. London: Allen and Unwin. An introduction to post-paradox logicism which avoids the complexities of the ramified theory of types. [i]

For an incisive appraisal of Russell's logical views

Gödel, K. 'Russell's Mathematical Logic'. In B&P. [d]

iii. Ramsey

Distinguishing between mathematical and semantic paradoxes, Ramsey defended foundations based on non-predicative type theory as against predicative type theory

Ramsey, F. 1925. 'Foundations of Mathematics'. In *Philosophical Papers*, edited by D. H. Mellor, Cambridge: Cambridge University Press, 1990. [d]

B. HILBERT

An alternative foundational programme was proposed by Hilbert. The underlying ideas are to be found in

Hilbert, D. 'On the Infinite'. Abridged in B&P.

——. 'The Foundations of Mathematics'. In Jan van Heijenoort, ed., *Frege and Gödel: two Fundamental Texts in Mathematical Logic*. Cambridge, Mass.: Harvard University Press, 1970. [d]

Digestible expositions of Hilbert's programme are hard to come by. Kreisel's paper in B&P is technical and assumes too much to count as an exposition. Stressing the connections between Hilbert's programme and logical positivism is

Giaquinto, M. 1983. 'Hilbert's Philosophy of Mathematics'. *British Journal for the Philosophy of Science* 34: 119-132.

C. GÖDEL'S INCOMPLETENESS THEOREMS

Gödel's incompleteness theorems effectively sunk the (very different) foundational outlooks of Russell and Hilbert, this is the standard view. Again there appears to be no good non-technical exposition of the matter. For Gödel vs. Hilbert something may be gleaned from Giaquinto (*op. cit.*).

For opposition to the standard view, (in defence of Hilbert's programme), see

Detlefsen, M. 1986. *Hilbert's Programme: an Essay on Mathematical Instrumentalism*. Dordrecht: D. Reidel.

For a defence of the standard view against Detlefsen's argument see

Auerbach, D. 1992. 'How To Say Things With Formalisms'. In *Proof, Logic and Formalization,* edited by Michael Detlefsen. London: Routledge.

For an intuitionist's reaction see

Dummett, M. 1963. 'The Philosophical Significance of Gödel's Theorem'. *Ratio* 5: 140-155. Reprinted in *Truth and Other Enigmas*. London: Duckworth, 1978.

D. WITTGENSTEIN

Wittgenstein still gets much attention. He favoured a kind of conventionalism, but had no worked out view. Yet his exploratory remarks can be stimulating.

Wittgenstein, L. 1978. *Remarks on The Foundations of Mathematics.* Translated by G. E. M. Anscombe. 3rd ed. Oxford: Blackwell.

The best work on his philosophical remarks on mathematics is

Wright, C. 1980. *Wittgenstein on the Foundations of Mathematics.* London: Duckworth.

Look also at the discussion of Wittgenstein's conception of mathematics in

Dummett, M. 1959. 'Wittgenstein's Philosophy of Mathematics'. *Philosophical Review* 68: 324-348. Reprinted in *Truth and Other Enigmas*. London: Duckworth, 1978, and his later reflections on that paper in

Dummett, M. 1993. 'Wittgenstein on Necessity: Some Reflections'. In *The Seas of Language.* Oxford: Clarendon Press.

E. INTUITIONISM

The originator of intuitionism was Brouwer. His views are set out in

Brouwer, L. 'Consciousness, Philosophy and Mathematics'. Reprinted in B&P.

The leading exponent of intuitionism among English-speaking philosophers is Dummett. Brouwer's view is inspired by Kant; Dummett's by Wittgenstein (his philosophy of language, that is). See

Dummett, M. 1973. 'The Philosophical Basis of Intuitionistic Logic'. Reprinted in B&P; and in in *Truth and Other Enigmas*. London: Duckworth, 1978. [d]

Quite helpful is the introduction and first chapter of

Dummett, M. 1977. *Elements of Intuitionism.* 2nd ed. Oxford: Clarendon Press, 2000.

While his latest statement of his views of mathematics can be found in

Dummett, M. 1993. 'What is Mathematics About?'. In *The Seas of Language.* Oxford: Clarendon Press.

See also

Hellman, G. 1989. 'Never say "Never"! On the Communication Problem between Intuitionism and Classicism'. *Philosophical Topics* 17: 47-67.

Velleman, Daniel. 1993. 'Constructivism Liberalized'. *Philosophical Review* 102: 59-84.

F. PREDICATIVISM

The mathematician Henri Poincaré was the source of the predicativism of *Principia Mathematica*, but he was unsympathetic to logicism. A good expression of his views is

Poincaré, H. 1963. 'The Logic of Infinity'. In *Mathematics and Science: Last Essays.* Translated by John W. Bolduc. New York: Dover.

For the view of Whitehead and Russell see ch. 2 of the Introduction to Vol. 1,

Whitehead, A. N., and B. Russell, 1910. *Principia Mathematica.* 3 Vols. 2nd ed. Cambridge: Cambridge University Press, 1925-1927.

Weyl begins to overcome the problem that stumped the authors of *Principia Mathematica* before being thrown off-track by Brouwer. See part II of Maniosu (op. cit.) especially Maniosu's introduction to that part. See also the section on Weyl in Feferman (op. cit.).

Feferman, S. 1998. 'Weyl Vindicated: *Das Kontinuum* Seventy Years Later'. In *In the Light of Logic.* Oxford: Oxford University Press. Not only an excellent account of Weyl's predicativist programme but also an exposition of Feferman's own work in bringing the programme to fruition. [t]

A relatively recent defence of predicativism without logicism is

Chihara, C. 1973. *Ontology and the Vicious Circle Principle.* Ithaca: Cornell University Press.

For a recent appraisal of Poincaré's views

Folina, J. 1991. *Poincaré and the Philosophy of Mathematics.* Basingstoke: Macmillan.

For a shift of emphasis which illuminates the philosophical motive for both constructivist schools see

Detlefsen, M. 1991. 'Brouwerian Intuitionism'. In M. Detlefsen ed., *Proof & Knowledge in Mathematics*. London: Routledge.

G. GÖDEL

Gödel's views are closer to Plato's than any other notable philosopher of mathematics.

Gödel, K. 'What is Cantor's continuum Problem?'. Reprinted in B&P. [d]

This is vital. To go further you should consult volumes two and three of

Collected Works: Kurt Gödel, edited by Solomon Feferman, et al., 3 Vols. Oxford: Clarendon Press, 1986.

For reports of Gödel's late views authenticated by the man himself see

Wang, H. 1974. *From Mathematics to Philosophy*. London: Routledge and Kegan Paul.

For criticism of Gödel's views see Ch.II of Chihara (*op. cit.*).

H. SET THEORY

There has been an increasing amount of work done on the question of what conception of set we should have, in the light of responses to the paradoxes. In addition to the works mentioned above, see

Boolos, G. 'The Iterative Conception of Set'. Reprinted in B&P.

Lewis, D. 1991. *Parts of Classes.* Oxford: Basil Blackwell. [d]

Hallett, M. 1984. *Cantorian Set Theory & Limitation of Size*. Oxford: Clarendon Press. [d]

Bigelow, J. 1990. 'Sets are Universals'. In *Physicalism in Mathematics,* edited by A. D. Irvine. Dordrecht: Kluwer Academic Publishers. [d]

I. MODERN PLATONISM: QUINE

Expressions of Quine's empiricist view of mathematics are scattered throughout his publications. You might start with

Quine, W. V. 1955. 'Posits and Reality'. In *Ways of Paradox: and other Essays.* New York: Random House, 1966.

The *locus classicus* of Quine's view, detailing his break with the logical positivists, is

Quine, W. V. 1954. 'Carnap and Logical Truth'. In *Ways of Paradox: and other Essays.* New York: Random House, 1966. Reprinted in B&P.

But also see references in

Parsons, C. 1985. 'Quine's Philosophy of Mathematics'. In L. E. Hahn, and P. A. Schilpp, eds., *The Philosophy of W. V. Quine*. La Salle, Ill.: Open Court.

Quine, W. V. 1985. 'Reply to Charles Parsons'. In L. E. Hahn, and P. A. Schilpp, eds., *The Philosophy of W. V. Quine*. La Salle, Ill.: Open Court.

Quine's indispensability view has been recently criticised in

Maddy, P. 1992. 'Indispensability and Practice'. *Journal of Philosophy* 89: 275-289.

Sober, E. 1993. 'Mathematics and Indispensability'. *Philosophical Review* 102: 35-57.

Quine's view has been extended and defended in

Resnik, M. 1997. *Mathematics as a Science of Patterns.* Oxford: Clarendon Press.

J. MODERN PLATONISM: NEO-FREGEANS

There has been renewed interest, in recent years, in applying Frege's 'context principle' to argue for some form of platonism.

Wright, C. 1983. *Frege's Conception of Numbers as Objects.* Aberdeen: Aberdeen University Press.

Dummett, M. 1991. *Frege: Philosophy of Mathematics.* London: Duckworth. Esp. Chs. 15-17.

Field, H. 1989. 'Platonism for Cheap? Crispin Wright on Frege's Context Principle'. In *Realism, Mathematics & Modality.* Oxford: Basil Blackwell.

Hale, B., and C. Wright. 1994. 'A *Reductio ad Surdum*? Field on the Contingency of Mathematical Objects'. *Mind* 103: 169-184.

Hodes, H. 1984. 'Logicism and the Ontological Commitments of Arithmetic'. *Journal of Philosophy* 81: 123-149.

Heck, R. 1992. 'On the Consistency of Second-order Contextual Definitions'. *Noûs* 26: 491-494.

K. MODERN ANTI-PLATONISM

Feferman argues that only a small part of mathematics is needed for science

Feferman, S. 1998. 'Why a Little Bit goes a Long Way: Logical Foundations of Scientifically Applicable Mathematics'. *In the Light of Logic.* Oxford: Oxford University Press. Ch.14. [t]

Field argues that science can dispense with numbers; taking Quine's epistemology as gospel he concludes that arithmetic is false

Field, H. 1980. *Science without Numbers*. Oxford: Blackwell.

For criticisms of Field's attempt see

Shapiro, S. 1983. 'Conservativeness and Incompleteness'. *Journal of Philosophy* 80: 521-530; reprinted in W. D. Hart, ed., *The Philosophy of Mathematics.* Oxford: Oxford University Press, 1996. [t]

An alternative approach to ‘fictionalism’ about maths

Papineau, D. 1993. *Philosophical Naturalism*. Oxford: Blackwell. Ch.5.

An alternative empiricist view is developed by Kitcher

Kitcher, P. 1983. *The Nature of Mathematical Knowledge*. Oxford: Oxford University Press.

Of relevance to the debate about Field’s fictionalism is the question of what interpretation of modality an anti-platonist can appeal to. On this topic look at the following

Putnam, H. 1967. ‘Mathematics without Foundations’. *Journal of Philosophy* 64: 5-22. Reprinted in *Mathematics, Matter and Method*, Philosophical Papers, Vol. 1. Cambridge: Cambridge University Press, 1975; also reprinted in W. D. Hart, ed., *The Philosophy of Mathematics.* Oxford: Oxford University Press, 1996.

Kessler, G. 1978. ‘Mathematics and Modality’. *Noûs* 12: 421-441.

Chihara, C. 1990. *Constructibility and Mathematical Existence*. Oxford: Clarendon Press. Pt.1.

Hellman, G. 1989. *Mathematics without Numbers: Towards a Modal-structural Interpretation.* Oxford: Clarendon Press.

Burgess, J., and G. Rosen. 1997. *A Subject with No Object: Strategies for Nominalistic Interpretation of Mathematics*. Oxford: Clarendon Press.

For the connection between these issues and Field’s fictionalism then look at these

Field, H. 1990. ‘Mathematics and Modality’. In G. Boolos, ed., *Meaning & Method*: *Essays in Honor of Hilary Putnam*. Cambridge: Cambridge University Press. Reprinted In *Realism, Mathematics & Modality.* Oxford: Basil Blackwell, 1989.

—— 1991. ‘Metalogic and Modality’. *Philosophical Studies* 67: 1-22.

Shapiro, S. 1993. ‘Modality and Ontology’. *Mind* 102: 455-481.

L. STRUCTURALISM

According to structuralists, mathematics is the science of patterns; they differ over what account of patterns they favour.

Benacerraf, P. 1965. ‘What Numbers Could Not Be’. *Philosophical Review* 74: 47-73. Reprinted in P&B.

Resnik, M. 1994. ‘What is Structuralism?’. In D. Prawitz, and D. Westerståhl, eds., *Logic and Philosophy of Science in Uppsala: papers from the 9th International Congress of Logic, Methodology, and Philosophy of Science*. Dordrecht: Kluwer. A useful short paper discussing relations between structuralism and Platonism re mathematical objects.

Resnik sets out and discusses difficulties for structuralism in his clear no-nonsense way in

Resnik, M. 1988. ‘Mathematics from the Structural Point of View’. *Revue Internationale de Philosophie* 42: 400-424.

For an early overview of his account, see also

Resnik, M. 1982. ‘Mathematics as a Science of Patterns: Epistemology’. *Noûs* 16: 95-105.

——. 1981. ‘Mathematics as a Science of Patterns: Ontology and Reference’. *Noûs* 15: 529-566.

——. 1997. *Mathematics as a Science of Patterns.* Oxford: Clarendon Press. This is his most recent statement.

Shapiro, S. 1989. ‘Structure and Ontology’. *Philosophical Topics* 17: 145-171.

——. 1983. ‘Mathematics and Reality’. *Philosophy of Science* 50: 523-548.

——. 1997. *Philosophy of Mathematics: Structure and Ontology.* New York: Oxford University Press..

See also

Philosophia Mathematica, Vol.4, May 1996. It has articles by Shapiro, Resnik, Hellman, Hale, MacLane (the famous algebraist), and Benacerraf. None but Hellman’s is too recherché to be worth studying (unless you are into the details of modal-structuralism).

Parsons, C. 1990. ‘The Structuralist View of Mathematical Objects’. *Synthese* 84:303-346. Reprinted in W. D. Hart, ed., *The Philosophy of Mathematics.* Oxford: Oxford University Press, 1996. This provides an overview and critique of all of the modalist and structuralist positions.

M. MATHEMATICS & KNOWLEDGE

If one combines Goldman’s (1967) view of knowledge with Field’s view of truth and reference one gets Benacerraf’s problem

Field, H. 1972. ‘Tarski’s Theory of Truth’. *Journal of Philosophy* 69: 347-375.

Goldman, A. 1967. ‘A Causal Theory of Knowing’. *Journal of Philosophy* 64: 357-372.

Benacerraf, P. 1973. ‘Mathematical Truth’. *Journal of Philosophy* 70: 661-679. Reprinted in B&P.

Field, H. 1989. ‘Realism, Mathematics & Modality’. In *Realism, Mathematics & Modality.* Oxford: Basil Blackwell. Sec. 2.

Maddy’s view is posed as an answer to Benacerraf

Maddy, P. 1980. ‘Perception and Mathematical Intuition’. *Philosophical Review* 84: 163-96. Reprinted in W. D. Hart, ed., *The Philosophy of Mathematics.* Oxford: Oxford University Press, 1996.

Maddy’s view is developed in her book

Maddy, P. 1990. *Realism in Mathematics*. Oxford: Clarendon Press.

Her view is criticised by

Chihara, C. 1990. *Constructibility and Mathematical Existence*. Oxford: Clarendon Press.
Lavine, S. 1994. *Understanding the Infinite.* Cambridge, Mass.: Harvard University Press.
Giaquinto, M. 1994. 'Epistemology of Visual Thinking in Elementary Real Analysis'. *British Journal for the Philosophy of Science* 45: 789-813.
——. 1996. 'Non-Analytic Conceptual Knowledge'. *Mind* 105: 249-268.
——. 1998. 'Epistemology of the Obvious: A Geometric Case'. *Philosophical Studies* 92: 181-204.
For a recent overview of findings in numerical cognition by a leader in the field see
Butterworth, B. 1999. *The Mathematical Brain.* Basingstoke: Macmillan.