MATH 2101 GEOMETRY PROJECTS, 2017

1. DIFFERENTIAL GEOMETRY

1.1. **The isoperimetric inequality in the plane.** References:

- Lecture notes (Felix Schulze, 2 pages) for a direct computational proof (only works in the plane),
- Steiner's proof (works only in the plane) assuming the existence of a smooth minimizer, 'A Historical Review of the Isoperimetric Theorem in 2-D, and its place in Elementary Plane Geometry' by Alan Siegel (the proof is on pages 4-5 but the whole paper is a nice survey).

  `http://www.cs.nyu.edu/faculty/siegel/SCIAM.pdf`
- Steiner's proof using symmetrization (works in any dimension), assuming the existence of a C^1 minimizer, slides of a talk by Treiberg available at

  `http://www.math.utah.edu/~treiberg/Steiner/SteinerSlides.pdf`

1.2. **The four-vertex theorem.** For a smooth closed curve C in the plane, there is the notion of 'curvature'. If p is a point of C, there is a best fitting circle S which touches C at p. If the radius of this circle is r, then the curvature $\kappa(p)$ of C at p is $1/r$. A *vertex* of C is a point at which κ is either a maximum or minimum.

The 4-vertex theorem then states that any closed curve has at least 4 vertices.

- Do Carmo: Differential Geometry of curves and surfaces
- Osserman, American Mathematical Monthly, **92**, 1985.

1.3. **Fenchel's Theorem.**

- Hsiung: A first course in Differential Geometry.

1.4. **Riemann sums for polytopes.**

- `https://arxiv.org/pdf/math/0608171.pdf`

The starting point is the idea that integrals are approximated by Riemann sums. For a closed interval, if one subdivides into N equal parts, then in fact if the function is smooth, the error has an asymptotic expansion in negative powers of N, with coefficients depending only upon the derivatives at the endpoints and the mysterious Bernoulli numbers. In the above paper, which is only 13 pages in length, Guillemin and Sternberg explain how this is proved and give a generalization to Riemann sums over polytopes, and the extent to which they can be approximated by integrals.

1.5. **Integral geometry.** Crofton's formula, Buffon's needle, and Buffon's noodle. To state Crofton's formula, let C be a smooth plane curve of length ℓ, and consider the set Λ of all straight lines in the plane. Crofton's formula states that that the subset of Λ of lines which meet ℓ has measure 2ℓ. Part of the challenge in this project is to understand all the terms in the previous sentence!

- Crofton's formula can be found in do Carmo's book on curves and surfaces, pp. 41–46.

An interesting application of this is estimating the length of a (perhaps very complicated) plane curve by counting its intersections with lines in various families of parallel lines (this is described in the reference above).

Buffon's needle problem is the following. Suppose you have a floor ruled with parallel lines distance d apart and you repeatedly drop a needle of lenght ℓ on this floor. What is the probability that that the needle crosses one of the lines?

Buffon's noodle is the same but you are dropping a wet, hence flexible, noodle of length ℓ on the floor. Then there is a remarkably simple formula for the expected number of line crossings by the noodle, depending only upon d and ℓ. Crofton's formula gives a way to derive this.

Buffon's Noodle is described in

- J F Ramaley, American Mathematical Monthly, Vol 76, No. 8 (1969), pp.916–918.

A rather comprehensive reference for this area is the book of Santalo on integral geometry (in the UCL library).

2. TOPOLOGY

2.1. **Scissors congruence and Hilbert's Third Problem.** Let P and P' be two polyhedra in 3-space. We say that P and P' are *scissors congruent* if there is a way of chopping up one of them into (a finite number of) smaller polyhedra, and reassembling these polyhedra to make the other one.

If P and P' are scissors-congruent, then it is clearly necessary that $\mathrm{vol}(P) = \mathrm{vol}(P')$. It was conjectured by Hilbert in 1900 that there might be pairs of tetrahedra of equal volume but not scissors-congruent.

This was proved by Dehn who constructed another 'invariant' $D(P)$ with the property $D(P) = D(P')$ whenever P and P' are scissors-congruent, and then explicitly constructed two polyhedra P and P' but with $D(P) \neq D(P')$.

- Aigner/Ziegler, Proofs from the book, pp. 45–51.
- Dupont, 'What is...a scissors congruence?', Notices of the American Mathematical Society, **59**, 2012.

2.2. **Brouwer's fixed point theorem and the ham sandwich theorem.**

- Reference: Aigner/Ziegler, Proofs from the book, pp. 131–133.

See also

`http://math.mit.edu/~fox/MAT307-lecture03.pdf`

This is the proof is via 'Sperner's Lemma'. An application is the so-called 'ham sandwich theorem'.

2.3. **Knot complements.** Suppose I take a knot, K, in $\mathbf{R}^3$ and I bore out a neighbourhood of this knot and throw it away; I obtain a 3-dimensional manifold, M. What is the fundamental group of M? It is called the *knot group* of G_K. We can find a presentation for this group starting from a picture of the knot K. This is called the Wirtinger presentation.

- Use the Wirtinger presentation to find presentations of knot groups for some simple knots (including the unknot U and the trefoil knot T).
- Can you prove that the groups G_U and G_T are not isomorphic? That would tell you that the trefoil knot is really knotted (not the unknot).
- Find the knot group of the square knot and the granny knot; show that these groups are isomorphic. I claim that these two knots are nonetheless different. How might you tell them apart?
- For bonus points, explain the proof that Wirtinger's algorithm gives a presentation for the knot group.

Resources:

- Hatcher, 'Algebraic topology.' Cambridge University Press (2002) - also freely available here:

  `https://www.math.cornell.edu/~hatcher/AT/ATpage.html`
- Rolfsen, 'Knots and links.' American Mathematical Society (1990).
- Knot atlas, `http://katlas.org`

2.4. **Seifert surfaces and the knot genus.** A knot is not the unknot if it doesn't bound a disc. In other words, the boundary of a disc is, by definition, unknotted. But more complicated surfaces (cylinders, punctured tori) can have knotted boundaries. If K is a knot and S is a surface with boundary equal to K then we say S is a *Seifert surface* for K. The genus of a knot is defined as the minimal genus of all its Seifert surfaces and is an important and difficult-to-compute knot invariant: although it's often easy to find Seifert surfaces, it's not always easy to tell if your surface minimises genus.

- Draw some Seifert surfaces for simple knots; learn about Seifert's algorithm for finding a Seifert surface.
- Show that the genus is additive under "connected sum" of knots (why does this mean that the unknot cannot be written as a connected sum of nontrivial knots?).
- Learn about the Seifert matrix and the Alexander polynomial. The degree of the Alexander polynomial gives a lower bound on the genus of a knot - why?

Suggested resources:

- Adams, 'The Knot Book.' American Mathematical Society (2004) for a nice introduction to surfaces with pictures
- Cromwell, 'Knots and links', Cambridge University Press (2004) Chapters 5-7 for the detailed theory of Seifert surfaces, Seifert matrices and the Alexander polynomial.

2.5. **Braids.** Fix n points $p_1, \ldots, p_n$ in the plane $\mathbf{R}^2$. An *n-strand braid* is a collection of n pairwise-disjoint paths $\gamma_1, \ldots, \gamma_n$ in $\mathbf{R}^3$ such that $\gamma_k(t) = (x_k(t), y_k(t), t)$ and $(x_k(0), y_k(0)) = p_k$, $(x_k(1), y_k(1)) = p_{\sigma(k)}$ for some permutation σ of $\{1, \ldots, n\}$. We consider two braids to be equivalent ('isotopic') if there is a family of braids interpolating between them. Braids on n strands up to isotopy form a group under concatenation. This is called the braid group B_n.

- One can interpret the braid group as the fundamental group of the configuration space of n points in the plane. Explain why (with pictures).
- We can take a braid and 'close it up' to obtain a knot or link. Alexander (see reference below) proved that any knot can be obtained as

a braid closure. Read and explain Alexander's paper (with worked examples!).

- Purely algebraically, the braid group is a very nice object. A presentation for this group was given by Artin. For example, the 3-strand group is
$$B_3 = \langle a, b \mid aba = bab \rangle.$$
Using Artin's presentation, Garside wrote a beautiful paper explaining some of the nice properties of the braid group, including a 'normal form' for any braid which allows you to see when two braids are the same (they're equivalent if and only if they look the same in normal form). Read Garside's original paper to learn more about the normal form of braids and explain it to us with examples.

Suggested resources:

- Hansen, 'Braids and coverings.' LMS Student Texts 18 (1989).
- Alexander, 'A lemma on systems of knotted curves.' Proc. Nat. Acad. Sci. U.S.A., Vol. 9, No. 3 (1923) 93–95.
- Garside, 'The braid group and other groups.' Quart. J. Math. Oxford (2) 20, (1967) 235–54.

Further Reference:

The book by Francis, 'A topological picture book', Springer, is a useful supplementary reference (for pictures!).

3. GROUPS AND GEOMETRY

3.1. **Finite subgroups of rotations.** In 'groups and geometry', you have met the group $SO(3)$ of rotations of three-dimensional euclidean space. This is an *infinite* group. It is natural to ask what are its finite subgroups. These are entirely classified and are related to classical symmetric objects, the regular polygons and the regular polyhedra. The goal of this project is to describe this classification. A possible reference is

- Elmer Rees, Notes on Geometry.

3.2. **Triangle groups.** A triangle group is generated by three elements a, b, c subject to the simple relations

$$a^2 = b^2 = c^2 = 1, (bc)^p = (ca)^q = (ab)^r = 1. \tag{1}$$

This is a completely algebraic definition, but it is very fruitful to study such group geometrically. It turns out that such groups can be realised by taking a, b, and c to be reflections in the sides of a triangle—but depending on the integers (p, q, r), this triangle may be spherical, euclidean, or hyperbolic.

A basic question, for example, is for which triples (p, q, r), this abstract group is *finite*. The condition is remarkably simple, but not at all obvious at first sight. In this project you will look at examples of triangle groups and how their relations to the two-dimensional geometries.

3.3. **Tilings of the plane.**

3.3.1. *Two-dimensional crystallography.* What are the regular tesselations of the plane, and what is the crystallographic restriction?

- (E.g. Coxeter, Introduction to Geometry, Chapter 4).

These three projects, on finite subgroups of $SO(3)$, triangle groups, and 2D crystallography, are interrelated in interesting ways.

3.3.2. *Penrose tilings.* What is a Penrose tiling? How does one see that such tilings do cover the plane, given that they cannot be generated by locally adding tiles to a partial tiling? In what sense are they non-periodic?

This project would need to go beyond the pictures and prove, for example, that there is no non-trivial translation preserving any given Penrose tiling.

A starting point are the notes of Rich Schwartz

`https://www.math.brown.edu/~res/MFS/handout7.pdf`

A comprehensive source on tilings is

- Tilings and patterns, by Branko Grünbaum and G. C. Shephard. W. H. Freeman and Company, New York, 1987, ISBN 0-7176-1193-1.

and this is in the UCL library.

3.4. **Tilings of the hyperbolic plane and surfaces of genus ≥ 2.** A closed orientable surface Σ of genus g is easily pictured as a two-dimensional sphere 'with g handles'. If $g = 1$ we have the surface of a torus (ring doughnut), and other baked goods (pretzels) correspond to $g > 1$.

By cutting open Σ we can view it as a polygon with $2g$ sides, identified in pairs. If $g > 1$, we can put this polygon in H^2, of just the right size, so that its translates precisely tile the whole space. In this way, Σ is equipped with a hyperbolic structure, whatever that means.

The goal of the project is to understand the above claims and undefined terms.