Project Titles for MATH0084
2022-23

Suggested projects for 2022-2023 are arranged here alphabetically by supervisor. Most projects have either prerequisite or suggested modules. **Please contact supervisors directly** if you would like to find out more information about a project or to register an interest in taking that project.

- **Dr Riaz Ahmad** r.ahmad@ucl.ac.uk

  Applied Stochastic Calculus for Finance and Medicine – Mathematical Theory and Computation (with Python).

  This project specification is suitable for up to two students to conduct a survey of stochastic differential equations applied to solving problems in:

  — derivative pricing;

  — infectious diseases and cancer modelling.

  Programming will be an important part of the project and willingness to further develop skills in Python is essential. There is flexibility for the student to propose direction of the study based on their current interests or planned work/postgraduate study.

  Prerequisites: MATH0056 (Methods 4); MATH0057 (Probability and Statistics); MATH0033 (Numerical Methods).

  Some initial reading:

  (1) Wiersema, U F; Brownian Motion Calculus. Wiley 1st Edn (2008);

  (2) Wilmott, P; Paul Wilmott on Quantitative Finance. Wiley 2nd Edn: 3 Volume Set (2006);

  (3) Vynnycky, E; White, RG; An Introduction to Infectious Disease Modelling. OUP (2010);

  (4) Better modelling of infectious diseases: lessons from covid-19 in China [link].

- **Dr Stephen Baigent** steve.baigent@ucl.ac.uk

  1. *Invariant manifolds of discrete-time dynamical systems*

     Many models that arise in theoretical ecology and evolutionary game theory have curves, surfaces, or more generally manifolds, that are left invariant by the dynamics. For models that have an invariant manifold that attracts all points, the model can be solved by restricting to the manifold, generally an easier problem. The first aim of the project will be to learn some key mathematical theory for showing when these invariant manifolds exist, and then to apply the theory to several well-known models from theoretical ecology. A second aim will be to find (hopefully new) ways of computing these invariant manifolds using a program such as Mathematica, Python or Matlab, and then use the computations to push the models to limits where the invariant manifolds lose smoothness and eventually disappear.

     Pre-requisites: Some experience of programming would be useful. No Mathematical Biology background is needed.

2. *Optimal habitat choice with travel costs*

   This project builds on one supervised 2 years ago. Imagine a large population consisting of several species that populate a fixed habitat. Each habitat has a range of resources and each species has food preferences, safety concerns, etc. How does the population fill out the habitat? This is an evolutionary game and the choice is usually a special kind of Nash equilibrium known as an evolutionarily stable state. If it actually costs individuals to move between sites, this complicates the problem: They may not move even if they would be 'fitter' in the new site if it costs too much in fitness terms to reach it. The theory for this is not so well known, and possibly not known for some models. The aim will be to formulate a new 2D partial differential equation model where there is not a finite number, but a continuum of species. The first task will be to set up the model and show that it makes sense (has meaningful solutions for reasonable scenarios) and the second task will be compute these solutions using finite-differences, finite elements, or similar, using Mathematica, Python, Matlab or a suitable pde solver, and explore cost-benefit trade-offs for different models.

   Pre-requisites: Some experience of programming would be useful. While the project includes some game theory ideas covered in the 2nd term module Evolutionary Games and Population Genetics (MATH0082), the project can be done independently.

- **Dr Costante Bellettini** c.bellettini@ucl.ac.uk

1. *Minimality of the Simons cone*

   Discovered to be a stable minimal hypersurface by J. Simons, this 7-dimensional cone in $\mathbb{R}^8$ was proved to be a minimizer of the area by Bombieri, De Giorgi and Giusti (about 50 years ago). The discovery of this area-minimizer with a singular point indicates that the minimizing problem has to be posed in a class of non necessarily smooth hypersurfaces (sets of finite perimeter in geometric measure theory, a field where measure theory and differential geometry merge). A singularity formation of this type does not arise in dimensions up to 6 and it is still mysterious nowadays what makes dimension 7 so special.

2. *Allen-Cahn energy and minimal surfaces*

   The Allen-Cahn equation is a second order elliptic semi-linear partial differential equation used to describe phase separation of a two-phase liquid (two components of a binary fluid spontaneously separate and form domains that are pure in each component). In recent years this PDE has had striking impact on geometric problems and active research is ongoing in this direction. The connection with geometry lies in the fact that the interface of separation between the two phases of the liquid tends to be locally area minimizing: the liquid tries to use as little area as possible to transition from one phase to the other. This feature of the Allen-Cahn energy has prompted, for example, a new and rather straightforward proof of the following fundamental result (originally proved in the 80s): in every closed Riemannian manifold there exists a closed minimal hypersurface. Minimal means that the mean curvature is everywhere zero - this is the geometric counterpart of the area minimization property. The lowest-dimensional example of a minimal hypersurface is a geodesic on a surface (e.g. an equator on a sphere).

3. *Other topics in geometric measure theory and in elliptic partial differential equations (Monge-Ampere equation, harmonic maps, etc).*

Pre-requisites for these projects: Analysis 4 (MATH0051). Any of Measure Theory (MATH0017), Linear Partial Differential Equations (MATH0070) and Differential Geometry (MATH0020) would be helpful.

- **Dr Christopher Birkbeck** c.birkbeck@ucl.ac.uk

  1. *Mathematics in Lean*

     Mathematical formalisation is a new area in-between mathematics and computer science, which seeks to turn theorems and proofs into code that computers can understand, allowing them to not only do numerical calculations, but also create logical arguments. . Lean is a specialized programming language that allows one to turn mathematics into such code. This turns maths into a computer game in which to pass a level you need to prove a theorem (see for example the "natural numbers game"). One of the main aims is to create a unified digital library (which in Lean in known as mathlib) of all of mathematics, with the hope of finding new links between areas of mathematics and perhaps finding new proofs by using machine learning techniques. This project involves learning how one uses Lean and then formalising piece of mathematics. Given that this area is very new, there are many undergraduate level results which have yet to be formalised, therefore it is likely that the resulting code could form part of Lean's mathlib. For example, there are many basic results in number theory which have yet to be formalised, and this could be one possible avenue, but there is plenty of scope to look at other areas depending on interests.

     Pre-requisites: Some coding experience in any computer language would be useful but not essential.

  2. *Modular forms*

     Modular forms are special analytic functions with many amazing properties. They are the glue between algebra, geometry, analysis, and number theory. They feature in many of the most important results in number theory, such as Fermat's Last Theorem. The most basic definitions are that of analytic functions which transform in a specific way under the action of 2x2 integer matrices with determinant 1. This project involves looking at either generalised versions on modular forms (such as p-adic modular forms or modular forms over number fields) or studying their applications in number theory and beyond (such as links with Elliptic curves, the Leech lattice, etc).

     Pre-requisites: MATH0104, MATH0034, MATH0035, MATH0036.

- **Prof. Steven Bishop** s.bishop@ucl.ac.uk

  Some events, such as car accidents and crime, can be considered as being rare. Some people never experience these, while others experience them more than once. How can we say whether their occurrence is just bad luck, associated with a random event, or whether there is more to it than that. For rare events, the usual statistical distributions do not fit the data. In an earlier paper, an index was introduced, called the Rare Event Concentration Coefficient (RECC), to measure the dispersion/concentration of events which have a low frequency but tend to have a high level of concentration. The method is based on a statistical mixture model. The project would be to review this work and then apply the technique to a new data set. It would require the student to have knowledge and skills in statistics and an interest in mathematical modelling.

Based on: Prieto Curiel, R. and Bishop, S., *A measure of the concentration of rare events*, Scientific Reports Vol 6, Article number: 32369 (2016).

- **Dr Christian Boehmer** c.boehmer@ucl.ac.uk

  1. *Continuum mechanics with microrotations*

     The aim of this project is to study elasticity theory in the presence of micro-rotations. This theory is known under a few different names like Cosserat elasticity or Micropolar elasticity. As a first step the candidate would have to become familiar with elasticity theory (linear and non-linear) and next include micro-rotations. Various routes could be explored ranging from more computational work using Mathematica or more analytical work which would involve the calculus of variations to study equations of motion.

     Pre-requisites: No particular prerequisites.

  2. *Modified theories of gravity with diffeomorphism non-invariance*

     The first part of the project is to study the variational approach to the Einstein field equations and looking at the original Einstein action, sometimes called the Gamma squared action, which is different from the Einstein-Hilbert action commonly used. This can be used to set up a modified theory of gravity with second order field equations similar to those found in other popular modified gravity models. Interestingly, this model is no longer diffeomorphism invariant in general. The main part of the project is about studying this model in some concrete situations like cosmology, spherical symmetry or the study of gravitational waves. There are many avenues that can be explored further.

     Pre-requisites: Mathematics for General Relativity (MATH0025)

  Pre-requisites: Please note that most of these projects require a good deal of programming in Mathematica. It is therefore essential that candidates have some programming background and are willing to invest effort into learning Mathematica.

- **Dr Robert Bowles** r.bowles@ucl.ac.uk

  1. *The impact of viscous effects on free surface flow*

     The project examines a range of viscosity-influenced free-surface flows, for example planar and axisymmetric flows, flow over obstacles, hydraulic jumps. Details can be discussed with the student. Some computational work is likely to be required.

     Prerequisites or to take concurrently: Real Fluids (MATH0077), Asymptotic Approximation Methods (MATH0078)

  2. *The flow emerging from a tap*

     This project analyses the flow of a Newtonian fluid as it emerges from a vertical circular pipe into the open air. Important physical effects such as the switch from no-slip to no-shear boundary conditions, the impact of surface tension and the inertia of the jet itself can be analysed and their interactions studied. Extensions to non-Newtonian fluids may be possible. Computational work is required.

     Prerequisites (co-requisite): Real Fluids (MATH0077), Asymptotic Approximation Methods

- **Prof. Erik Burman** e.burman@ucl.ac.uk

    1. *Inverse boundary value problem for elastodynamics*

       In this project we will consider the equations of elastodynamics and explore numerically how the solution can be reconstructed when the data on part of the domain boundary is unavailable. To compensate for this lack of data we assume that some additional measurements are available, either in the bulk domain or on the boundary. The package FreeFEM++ will be used for the numerical computations.

       Pre-requisites: Variational Methods for PDEs (MATH0092)

    2. *Nonlinear viscosities for conservation laws*

       We will consider the Burgers equation in one space dimension and study how the addition of artificial viscosity can improve the behaviour of computational methods. In particular we will compare the effect of linear and nonlinear diffusivities close to and far away from discontinuities in the solutions

       Pre-requisites: Numerical Methods

- **Dr Cecilia Busuioc**. cecilia.busuioc@ucl.ac.uk

    1. *Periods of Modular Forms*

       The spaces of modular forms have been of interest to number theorists because they exhibit natural rational structures. In MATH0104, we saw that the vector space of modular forms of a given weight for the full modular group is spanned by modular forms with rational Fourier coefficients and its finite-dimensionality led to interesting identities with a wide-range of applications. The theory of Eicher-Shimura provides us with another rational structure coming from the periods of modular forms. The purpose of this project is to first understand the main theory and possibly look at some applications (e.g. in relation to binary quadratic forms, zeta-functions associated to real quadratic fields) and then to study some recent surprising results of D. Zagier and collaborators who show that once one assembles the Hecke eigenforms and their suitable period polynomials into a generating function, the result is a product of well-known theta functions. One could then further explore some consequences of this identity, such as recovering the Fourier coefficients of the Hecke eigenforms in question from the given identity, which the authors were only able to show for levels 2,3, and 5.

       Recommended pre-requisites: MATH0035(Algebraic Number Theory), MATH0104 (Modular Forms)

    2. *Modular Curves, Regulators of Siegel Units and Applications*

       In Number Theory, it is a classical approach to associate to an object of arithmetic significance an L-function defined by an Euler product encoding local information which then one hopes to relate to global, geometric objects. Conjectures of Zagier and Boyd are such examples in the case of an elliptic curve defined over the rationals. Recent work of F. Brunault gives us explicit formulas of regulators of Siegel units (these are units in the function field of a modular curve, which is the corresponding algebraic curve obtained from the quotient of the complex upper half plane by the action of a congruence subgroup) as Mellin Transforms of certain Eisenstein Series of weight 1, which can be used to provide numerical examples of the conjectures mentioned above. The goal of this project is to study Brunault's paper and possibly compute further numerical examples of the conjectural formulas.

       Recommended pre-requisites: MATH0036 (Elliptic Curves), MATH0104 (Modular Forms)

3. *Cyclotomic Fields and Iwasawa Theory*

   Recommended Pre-requisites: MATH0021(Commutative Algebra), MATH0022 (Galois Theory), MATH0035(Algebraic Number Theory)

4. *Other topics in Algebraic Number Theory and Arithmetic Geometry*

- **Dr Shane Cooper** s.cooper@ucl.ac.uk

  1. *Mathematical approach to innovative composite material design*

     In this project we shall study solutions of second-order partial differential equations with rapidly oscillating coefficients using asymptotic analysis. The equations of interest arise from mathematical models for the behaviour of modern advanced man-made composite materials.

- **Dr Matthew Crowe** m.crowe@ucl.ac.uk

  1. *Solutions of Monge-Ampere equations*

     Monge-Amprere equations are a type of nonlinear PDE which commonly arise in oceanic and atmospheric problems when considering the vorticity of the system. The nonlinear nature of these equations makes finding numerical solutions difficult. The aim of this project is to find a way of transforming these equations to a linear form which can be solved numerically and to implement the transformation and solution. The transformation may give some insight into when solutions exist. This project is open ended with scope for a more 'theoretical' Analysis-based approach or a more 'practical' numerical approach.

     Pre-requisites: These vary with the project direction. Any of; Methods 4 (MATH0056), Fluid Mechanics (MATH0015), Analysis 4 (MATH0051), Multivariable Analysis (MATH0019) and Linear PDE (MATH0070) may be useful.

  2. *Internal waves in a sloping channel*

     Internal waves occur in fluids where the density changes with depth and are common features of the ocean and atmosphere. When analytically examining waves in a channel, we often assume the walls are vertical allowing simple boundary conditions to be applied. The aim of this project is to use an asymptotic approach to consider the effects of sloping side-walls and bottom and to derive a single (KdV-like) equation to describe the evolution of the wave amplitude. This project is primarily analytical though there is scope for numerical simulations if desired.

     Pre-requisites: Fluid Mechanics (MATH0015). Geophysical Fluid Dynamics (MATH0024) would be useful.

- **Dr Mohit Dalwadi** m.dalwadi@ucl.ac.uk

  1. *Fundamental models of multiscale mass and fluid transport*

     Multiscale problems of mass transport are ubiquitous in physical applied mathematics. Applications include fluid transport in tumours, membrane filtration, nutrient delivery to plant roots in soil, salt transport in sea ice formation, and many more. In this project, the student will review basic partial differential equation models for multiscale mass and fluid transport, and go on to investigate asymptotic solution structures when regions involving different dominant transport mechanisms are coupled together. There are also opportunities - but no requirements - to write numerical simulations in this project.

Pre-requisites: Advanced Modelling Mathematical Techniques (co-requisite), Asymptotic Approximation Methods (co-requisite), Mathematical Methods 5 [or a willingness to learn aspects of each].

2. *Cryopreservation*

   Cryopreservation technology is used for applications involving fertility, tissue transplantation, and the protection of endangered species. Mathematical models can be used to understand how to reduce cell damage in this process. Since freezing and melting involve transitions between ice and water phases, mathematical models of this process can involve solving partial differential equations with moving boundaries, where the position of the domain boundary must be determined as part of the solution. In this project, the student will review basic mathematical models for freezing, then investigate how adding cryoprotective chemicals can reduce cell damage in cryopreservation. There will be opportunities to use both asymptotic and numerical methods in this project.

   Pre-requisites: Advanced Modelling Mathematical Techniques (co-requisite), Asymptotic Approximation Methods (co-requisite), Mathematical Methods 5 [or a willingness to learn aspects of each].

3. *Decontaminating chemical agents*

   When harmful chemical agents are spilled it can be incredibly harmful to people and the environment, so it is vital to be able thoroughly decontaminate affected areas. Mathematical models can be used to understand how to choose appropriate cleansers when confronted with novel chemical agents in the field. Such models typically involve solving partial differential equations with moving boundaries, where the position of the domain boundary must be determined as part of the solution. In this project, the student will review basic mathematical models for chemical decontamination, then explore more complex set-ups, such as emulsions of agent and cleanser. There will be opportunities to use both asymptotic and numerical methods in this project.

   Pre-requisites: Advanced Modelling Mathematical Techniques (co-requisite), Asymptotic Approximation Methods (co-requisite), Mathematical Methods 5 [or a willingness to learn aspects of each].

- **Dr Ben Davies** ben.m.j.davies@ucl.ac.uk

  1. *Automating assessments for tertiary mathematics in STACK*

     STACK is a new programming language developed to aid learning designers in the automation of assessment in mathematics. This tool, housed in Moodle and built on Maxima, allows designers to randomise question variants, personalise feedback based on student input, and develop intricate pedagogic sequences based on sophisticated mathematical code. In this project, the student will be expected to produce a suite of mathematically sophisticated question models, and to develop a series of design principles informing future design. An interest in education and mathematical pedagogy is a must, and previous coding experience is highly recommended.

  2. *Proof comprehension and the role of summarising in mathematical argumentation*

Mathematicians rarely write out a mathematical proof 'in full', with explicit mentions of axioms defining their operational space and without any implicit warrants linking statements. Rather, we rely on shared conventions to abbreviate their mathematical arguments to only the most salient derivations. As such, 'summarising' is central to mathematical practice. In this project, the student will take a deep theoretical look at the activity of summarising across disciplines, and will apply these ideas in the realm of mathematical proof. The project lies at the intersection of mathematical, (experimental) philosophy and statistics, and could follow many different paths according to the student's interest.

3. *Spread of mathematical knowledge*

   The mathematical community shares knowledge via a variety of different media. Despite the innovations of the 21st century, peer-reviewed journals and chalk-and-talk seminars remain pivotal in the spread of mathematical knowledge. In this project, the student will study this spread by taking an active role within the community of research mathematics. The student will be expected to attend a seminar series of their choosing, and to document the experience from the position of a 'participant-observer'. This ethnographic investigation will focus both on the mathematical insights gained, and the ways in which these new insights came about. This project will suit a confident, self-reflective individual, capable of detailed note-taking and deep reflection on their own learning process.

- **Dr Alejandro Diaz** alex.diaz@ucl.ac.uk

1. *Calibration of computer models through History Matching*

   Computer models are essential in modern science to study the behaviour of complex systems. The reliability of these models depends on how well they are calibrated to data. However, some models are extraordinarily expensive, and a reduction of the input space prior to calibration is needed to mitigate the computational cost. History Matching is a technique designed for this, by carefully sampling the input space, given all the known sources of uncertainty. The aim of this project is to design strategies to make model calibration more efficient, by either establishing new theoretical properties or defining new sampling algorithms. The project requires familiarity with basic probability theory and proficiency in coding. Matlab, Python and R are suitable programming languages.

- **Dr Vladimir Dokchitser** v.dokchitser@ucl.ac.uk

1. *The inverse Galois problem and elliptic curves*

   It is a long-standing unsolved problem whether every finite group is the Galois group of a polynomial over the field of rational numbers. The aim of the project is to make a survey of some known results on the problem in general, and then describe in detail how to use elliptic curves to construct extensions with certain Galois groups.

   Prerequisites: Galois Theory (MATH0022) and Elliptic Curves (MATH0036) are essential. Algebraic Geometry (MATH0076), Representation Theory (MATH0073) and Algebraic Number Theory (MATH0035) are desirable.

2. *Class field theory*

   Class field theory is one of the greatest achievements of early 20th century number theory. In effect, it allows one to replace the study of ideal class groups with Galois theory (via 'class fields'), and to describe all extensions of number fields with abelian Galois groups using class groups and their generalisations ('ray class groups'). The aim of the project is to present the main statements of global class field theory and to illustrate them with applications and numerical examples. The student should focus on the classical formulation in terms of moduli and ray class groups and class fields, rather than the more abstract approach using adeles and ideles.

Prerequisites: Galois Theory and Algebraic Number Theory are essential.

- **Prof. Gavin Esler** j.g.esler@ucl.ac.uk

1. *Dam breaks in a rotating fluid*

   Two classic solutions in the fluid dynamics of shallow fluid layers are the dam break solution of Ritter (1892) and the geostrophic adjustment solution of Gill (1976). The first is a nonlinear solution when rotation is absent, and the second is a linear solution when rotation is present. This project will investigate what happens when the flow is rotating and the (partial) dam break is nonlinear, using the above solutions as references. The option of using either a numerical computation approach or an analytical approach, or both, will be available depending on student interest and background.

   Prerequisites: Fluids MATH0015. Geophysical fluids MATH0024 and Asymptotic methods MATH0078 are helpful but not necessary.

2. *Planetary jets driven by random forcing*

   A classic problem in geophysical fluids is the understanding the formation of planetary-scale jets (e.g. those seen on Jupiter) in models forced by a stochastic process (representing small scale convective processes or instabilities). Recent breakthroughs in this area mean that in a relevant idealised limit, the effect of the stochastic forcing can be represented by a deterministic term which depends on certain local flow properties. The project will aim to review this work and explore new jet-like flows, with the option of using either a numerical computation approach or an analytical approach, or both, being available depending on student interest and background.

   Prerequisites: Fluids MATH0015. Geophysical fluids MATH0024, as well as having encountered stochastic processes in at least one course, is helpful but not necessary. Asymptotic methods MATH0078 could also be helpful.

- **Dr Luis Garcia Martinez** luis.martinez@ucl.ac.uk

I am happy to discuss a variety of projects in number theory or algebra. Some examples are below.

1. *Elliptic curves and the class number problem*

   An interesting problem in algebraic number theory is how to explicitly construct extensions of a number field with abelian Galois group. For certain number fields this problem can be solved using elliptic curves. The goal would be to understand this construction and some of its applications.

   Prerequisites: Elliptic Curves (MATH0036). Recommended: Algebraic Number Theory (MATH0035).

2. *Arithmetic of Dynamical Systems*

   A diophantine equation is a polynomial equation that is to be solved in integers. In recent years techniques from the theory of dynamical systems have been used to obtain non-trivial results about diophantine equations. The project would be to understand some of these results, focusing on the relation between dynamical systems and heights.

   Prerequisites: Elliptic Curves (MATH0036). Recommended: Algebraic Number Theory (MATH0035).

- **Dr Selim Ghazouani** selim.ghazouani@gmail.com

  1. *Lorentzian manifolds*

     Lorentzian geometry is a generalisation of Riemannian geometry that is the conceptual framework for Einstein's relativity. While the theory is formally very similar to its Riemannian counterpart, it is a very different world altogether. For instance, not every manifold carries a Lorentzian structure. In this project, the student will study the interplay between the topology of manifolds and the Lorentzian geometry, starting with the following question: which two and three-dimensional manifolds carry a Lorentzian structure?

     Pre-(or Co-)requisite: MATH0072 Riemannian Geometry

  2. *Generic dynamics*

     A dynamical system is the datum of a transformation of a space (be it a homeomorphism or a differential equation) which determines the evolution of a point as time goes by. They are the mathematical formalisation of many a physical phenomenon, such as the evolution of the solar system or a gas particle moving freely within a box.

     This project will centre around the following question: what does a typical dynamical system look like? Mathematicians have come up with many different examples of systems evolving in qualitatively drastically different ways, but somehow experience shows that only a handful of them can actually be observed in nature. In particular we will discuss formal conjectures of Smale from the 70s putting forward a conceptual explanation for this phenomenon, and potentially more recent developments in the field of generic dynamics.

- **Dr Andrew Gibbs** andrew.gibbs@ucl.ac.uk

  Approximating integrals on fractal domains

  A quadrature rule is an approximation of a definite integral via a weighted sum of samples of the integrand. N-point Gaussian quadrature rules enjoy the 'exactness' property: their approximation is exact for polynomials of degree $2N-1$ (or less). These rules can be constructed on fractal subsets of $[-1,1]$, for example Cantor Sets, with respect to an appropriate measure.

  Much less is known about the construction of 'exact' N-point quadrature rules for subsets of $[-1,1]^M$, when $M>1$. These are typically referred to as 'cubature rules'. A common approach is to consider the N weights and N nodes as the unknown solution to an $2N$-dimensional non-linear system. In practice, this system is severely ill-conditioned, and must be solved numerically using high-precision arithmetic. This process is computationally expensive, but once the weights and nodes are computed, they give us fast and accurate approximations to integrals with smooth integrands.

In this project, the student will apply these techniques to construct cubature rules on fractals. These rules can then be incorporated into a wider software project, solving integral equations on fractal domains.

Prerequisites: Some programming experience in Matlab, Python or similar

- **Dr Luca Grieco** l.grieco@ucl.ac.uk

  1. *Strategic allocation of protective equipment for disaster preparedness*

     CORU has developed an analytical framework to help the UK Government make decisions on the stock of protective equipment to be held by ambulance services and hospitals for better preparedness to accidents involving the release of chemical, biological, radiological or nuclear (CBRN) materials [link].

     The aim of this project will be to define a more accurate method for estimation of the above stocks as well as replenishment strategies while accounting for demand variability, process variability and resource sharing policies. The choice of the specific CBRN event(s) and resources of interest will be discussed and agreed with the interested student. The approach will most likely involve a combination of mathematical optimisation and stochastic simulation. The student will explore the scientific and grey literature about existing approaches regarding the problem of interest, formulate a mathematical model tackling a relevant research question, implement it using appropriate software, analyse it to assess its usefulness and identify challenges regarding applicability of the approach in real life.

     Pre-requisites: some good knowledge of any programming language; depending on the approach followed, familiarity with optimisation techniques and probability theory would be advantageous; essential will be the desire to explore methodologies that might not have been covered in the modules attended so far.

- **Dr Mahir Hadzic** m.hadzic@ucl.ac.uk

  1. *Blueshift instability in General Relativity*

     The goal of this project is to understand the formulation of Strong Cosmic Censorship by R. Penrose and the associated blueshift effect originating from the work of Simpson and Penrose in early 1970s. Time permitting, we will discuss some of the more recent developments towards rigorous mathematical proofs of the Strong Cosmic Censorship. Basic knowledge of differential geometry and mathematical analysis are expected.

  2. *Wave equations outside obstacles*

     We consider the wave equation outside a compact obstacle. The goal is to understand the decay-in-time properties of the solution assuming suitable boundary conditions on the boundary of the obstacle. We shall consider the Dirichlet, the Neumann, and the Robin boundary conditions. Our starting point is the seminal work of Morawetz from 1960's which relies on the so-called multiplier / vector-field method.

     Prerequisites: Analysis 4 (MATH0051), Recommended: Multivariable Analysis (MATH0019).

- **Prof. Rod Halburd** r.halburd@ucl.ac.uk

1. *Riemann theta functions and differential equations*

   Riemann theta functions are entire functions of several complex variables defined by certain very simple rapidly converging series that play a central role in the theory of Riemann surfaces. In the one variable case they play a prominent role in the theory of elliptic functions, which are natural generalisations of trigonometric functions. They depend on a number of parameters, which often have a natural geometrical interpretation in terms of Riemann surfaces. They have many symmetries and satisfy a wide range of identities. As well as their intrinsic beauty and importance in the theory of Riemann surfaces, they are important special functions that can be used to give exact solutions to many problems, especially certain differential equations.

   Many of the PDEs that arise in the theory of solitons have infinitely many solutions given in terms of Riemann theta functions. Such equations have applications in water waves (e.g. the Korteweg-de Vires, Kadomtsev-Petviashvili and nonlinear Schrödinger equations), geometry (e.g. the sine-Gordon equation) and general relativity (e.g. the Ernst equation) etc. They are also related to the so-called finite-gap solutions of certain linear differential operators, which have important applications in quantum mechanics. Several well-known ODEs also have solutions in terms of theta functions, including some special cases of the equations of motion for a spinning top and the equations for geodesics on certain surfaces.

   Even when we set all the independent variables in our theta function to zero we still have some very interesting functions called thetanulls in which we consider the theta functions as functions of their parameters only. Thetanulls are related to modular forms and provide solutions to some equations first written down by Ramanujan as well as equations used to define special systems of orthogonal coordinates. They also provide some solutions to the self-dual Einstein equations.

   After learning some basic theory, this project can go in several directions. One is to explore in detail one or more application, such as those listed above. Another direction would be to explore some identities recently written down for higher-dimensional thetanulls and explore them as interesting differential equations.

2. *Topics in integrable systems*

   Integrable systems are equations (e.g. differential or discrete) for which the solutions have far more structure than one could expect generically. They can in some sense be considered to be exactly solvable. One possible project would be to look at some recently discovered (apparently) integrable differential-delay equations and to look for new examples of such equations. Another would be to study the self-dual Einstein equations.

3. *Analysis over the quaternions*

   There have been several attempts to create a satisfactory analogue of complex analysis for functions of a single quaternionic variable. We will explore a fairly recent development: the theory of slice-regular functions. Beyond understanding this theory, the aim of this project is to attempt to extend several theorems from complex analysis and/or the theory of special functions to this setting.

Pre-requesite for the above projects: Complex Analysis (MATH0013)

- **Dr Betti Hartmann** ucahbha@ucl.ac.uk

  1. The student would be working on a project related to nonlinear effects in classical field theory. The resulting coupled differential equations need (usually) to be solved numerically. The

student would be provided with tools to do so. Being able to program in FORTRAN, or willing to learn, would be useful. Possible projects of current interest could be:

a) Holographic superconductors with competing order parameters

b) Black holes and compact objects in extended gravity models

2. Black holes and other compact objects are accreting matter from their environment which leads to observable and quantifiable effects. The student would be investigating accretion processes around compact objects such as black holes and neutron stars by using large scale simulations and studying the observable effects. Good numerical skills would be very useful.

- **Dr David Hewett** d.hewett@ucl.ac.uk

Analysis on fractals

Fractals are beautiful and remarkable geometrical objects with structure on every lengthscale. Unlike with a smooth curve or surface, which appears “flat” when you look closely enough, with a fractal, no matter how far you “zoom in”, you still find geometrical structure. Some fractals, such as the middle third Cantor set, even exhibit perfect “self-similarity”, being a union of a certain number of scaled copies of themselves. Fractals provide examples of sets with non-integer dimension, and have many surprising properties - for example, the Koch snowflake has finite area but infinite perimeter! In this project the student will learn some basic results from the theory of fractal geometry, then go on to investigate questions relating to analysis on fractals, focussing for instance on:

1. Function spaces on fractals, e.g. Sobolev/Besov spaces, wavelet decompositions, trace operators. How to classify functions defined on fractals in terms of their “smoothness” (continuity, differentiability etc), when the underlying fractal set is highly non-smooth? Here there are open questions relating to the approximation of functions on fractals by piecewise polynomials.
2. Invariant measures and integration on fractals. Under certain assumptions it’s possible to define “invariant” measures on fractals that respect the self-similar structure of the fractal. One can then define integrals with respect to these measures, and there are interesting open questions relating to the numerical approximation of such integrals when the integrand is singular (i.e. blows up to infinity at certain points).

Prerequisites: MATH0051 Analysis 4, plus at least one of MATH0017 Measure Theory, MATH0018 Functional Analysis, MATH0070 Linear PDEs.

- **Dr Duncan Hewitt** d.hewitt@ucl.ac.uk

1. *Modelling coughing*

   Coughing - that is, a short pulse of strong air flow - is an efficient means of clearing airways, mobilising and transporting the mucus that lines the channel walls. Various illnesses, most notably cystic fibrosis, can affect mucus rheology, and affect the efficiency of the cough as a means of airway clearance. This project will explore modelling of the mechanics involved in a cough, focussing on how the mucus rheology can affect this.
2. *Propagation of water below ice sheets*

Understanding the mechanics of glaciers and ice sheets is important for predicting the impact of climate change, and the conditions at their base play a key role in controlling the motion of the overlying ice. Transmission of pressure and transport of water through the porous sediment or 'till' below glaciers affects the drag on the base of ice sheets. This project will review and develop mathematical models to describe fluid motion through this sediment layer, and explore their implications in cases of varying till permeability or topography.

3. *Fluid flow in deformable media*

   The coupling of solid deformation (which may be elastic or plastic in nature) and fluid flow is observed every time you squeeze out a kitchen sponge. Numerous industrial and household applications involve this phenomenon, from paper making to coffee pressing. This project will explore modelling of this behaviour, focussing in particular on heterogeneity in the stress or material properties of the sample being squashed.

All projects require knowledge from the methods courses, and all involve some fluid mechanics. Interested students are advised to take / have taken Real Fluids (MATH0077) and to take Industrial and Geological Fluids (MATH0106).

- **Dr Richard Hill** r.m.hill@ucl.ac.uk

1. *Topics in Number Theory*

   Prerequisites: the exact prerequisites will depend on which topic choosen, but you should have taken at least three of the modules Number Theory (MATH0034), Algebraic Number Theory (MATH0035), Elliptic Curves (MATH0036), Prime Numbers and their Distribution (MATH0083) by the end of the third year.

- **Prof. Francis Johnson** f.johnson@ucl.ac.uk

I offer a wide variety of projects in and around the areas of Algebraic Topology, Homological Algebra, Group Representation Theory and Discrete Subgroups of Lie groups. Typically these might include :

- Fibre bundles and spectral sequences;

- Lefschetz complexes and Poincaré duality;

- Projective modules and Algebraic K-Theory;

- Projective resolutions and the syzygetic approach to module cohomology;

- Representation theory of finite groups over $\mathbb{Q}$ and $\mathbb{Z}$;

- Borel density and Mostow rigidity.

- **Dr Nikoleta Kalaydzhieva** n.kalaydzhieva@ucl.ac.uk

The polynomial Pell equation

For a given non-zero positive integer D, which is not a square, we define Pell's equation to be $x^2 - Dy^2 = 1$, and is classically solved in positive integers x and y. Moreover, we know that solutions always exist and there are infinitely many of them. In this project we would try to better understand the polynomial Pell's equation, where for a given $D(t) \in \mathbb{C}[t]$, we try to find polynomials with complex coefficients $x(t),\ y(t)$. Do we always have solutions as in the classical case, and if so how many? We can also change our coefficient space and ask how that would change our problem.

Prerequisite: MATH0034 (Number Theory)

- **Dr Ilia Kamotski** i.kamotski@ucl.ac.uk

  1. *Topics in homogenisation theory*
     Prerequisite: Linear Partial Differential Equations (MATH0070)

- **Dr Nikon Kurnosov** n.kurnosov@ucl.ac.uk

  I am happy to discuss potential projects in algebraic and differential geometry, not limited to the list below.

  1. *Complex surfaces*
     There are many different complex surfaces and we will discuss the ways how one could classify them.

     Prerequisites: Multivariable Analysis (MATH0019), Differential Geometry (MATH0020), Algebraic Topology (MATH0023) or Topology and Groups (MATH0074)
  2. *Pell's equation and automorphisms*
     In this project we shall explore the automorphisms of certain class of manifolds. On the geometric side we will think about "rotations" of manifolds, on the algebraic point of view we will work with the lattices and the solutions of certain kind of arithmetic equation - Pell's equation, which appears to be very important in many areas of mathematics.

     Prerequisites: Differential Geometry (MATH0020), Algebraic Topology (MATH0023) or Topology and Groups (MATH0074)
  3. *Group theory in physics and chemistry*
     The project is about understanding the deep interactions between group theory and the basic chemical and physical properties of molecules and crystals. We will consider some applications of representation theory to spectra and to the conducting properties of materials.

     Prerequisites: Topology and Groups (MATH0074)

- **Dr Michal Kwasigroch** m.kwasigroch@ucl.ac.uk

  1. *Mathematical models of quantum spin liquids* A quantum spin liquid is an array of microscopic magnetic moments that remain strongly fluctuating, like a liquid, down to absolute zero temperature. Its state cannot therefore be easily parametrised in terms of the moment orientations. It turns out that the most natural description is in terms of 'fractions' of the magnetic moments. These natural degrees of freedom cannot be built from the microscopic constituents! The 'fractions' also carry a topological charge and interact with a background gauge field. The project will investigate and construct models for the dynamics of quantum

spin liquids, which involves the propagation of the above topological defects in the background of gauge fields. Different analytical and numerical approaches can be taken in this project such as approximate or numerical solutions of many-body equations of motion and Monte Carlo methods. Knowledge of quantum and statistical mechanics would be advantageous but not essential.

[1] C. Broholm et al., Science 367, 6475 (2020).

[2] L. Savary and L. Balents, Rep. Prog. Phys. 80, 016502 (2017).

2. *Mathematical models of quantum critical points*

   Fluctuations have a huge influence on the state of matter. They can even become infinitely strong if the external conditions are just right. Such conditions are known as quantum critical points and describe an amazing fractal state, where infinite fluctuations give birth to novel forms of matter. This is why quantum critical points are the matter analogue of stem cells. A particularly fruitful type of matter 'stem cell' is the ferromagnetic quantum critical point. The project will explore some of the possible mechanisms behind the formation of ferromagnetic quantum critical points. Different analytical and numerical approaches can be taken in this project such as self-consistent approximations, asymptotic methods and perturbation theory. Knowledge of quantum and statistical mechanics would be advantageous but not essential.

   [1] P. Coleman and A. J. Schofield, Nature 433, 226 (2005). [2] B. Shen et al., Nature 579, 51–55 (2020).

3. *Mathematical models of quantum fluids*

   A quantum fluid is characterised by long-range quantum coherence or entanglement between its distant parts. Therefore, the fluid has to be described by a complex wavefunction rather than a velocity field. This gives rise to many exotic phenomena such as quantisation of vorticity in rotating superfluids. Percolation is the phenomenon by which a fluid moves through a network that has some of its links removed. There is sometimes a non-zero critical link density, knows as the percolation threshold, below which the fluid is completely unable to traverse the network. The project will explore percolation in the context of quantum liquids and try to address the question of whether a percolation threshold can exist for a quantum liquid but not for a classical one. Different analytical and numerical approaches can be taken in this project such as self-consistent approximations, variational ansatze, Monte Carlo methods or exact diagonalisation of large sparse matrices. Knowledge of quantum and statistical mechanics would be advantageous but not essential.

   [1] Thomas Vojta and Jörg Schmalian, Phys. Rev. Lett. 95, 237206 (2005). [2] C. Zhang and B. Capogrosso-Sansone, Phys. Rev. A 98, 013621 (2018).

4. *Stochastic PDEs and thermalisation of quantum systems*

   The project will first review some of the generalisations of the Langevin equation, a stochastic PDE also used to model financial phenomena, to quantum systems. The project will then simulate the evolution of a quantum system in contact with an environment using one of these generalisations. Of particular current interest here, is the interplay between noise, friction and quantum entanglement, which the project will investigate. Entanglement presents many unique challenges to simulating quantum systems because of the exponential growth of the number of degrees of freedom with system size. A mixture of variational and Monte Carlo approaches will be used in an attempt to overcome or circumvent these challenges and simulate the stochastic equations. The project will also explore, time permitting, whether any lessons learned from the application of the Langevin equation to quantum systems can be transferred to financial systems.

[1] F. Reif, Fundamentals of Statistical and Thermal Physics, McGraw Hill New York (1965). (see section 15.5 Langevin Equation)

[2] K. Kanazawa et al., Phys. Rev. E 98, 052317 (2018).

[3] G. W. Ford and M. Kac, Journal of Statistical Physics 46, 803 (1987).

- **Dr Lars Louder** l.louder@ucl.ac.uk

  1. *Stallings' theorem on groups with infinitely many ends and groups of cohomological dimension 1*
  2. *Diophantine problems in the free group, or 10,000 ways to write a commutator*

     The word w=uuvuvvUUVUVV in the free group ¡u,v¿ can be written as a commutator in two essentially distinct ways: w=[uuvuv,vUU]=[uuvu,vvU] (check it!). It turns out that any commutator can be written, up to some natural equivalence which comes from the topology of compact surfaces, in only finitely many ways, and up to a slightly coarser, but still natural, equivalence, in at most 10,000 ways.

     The aim of the project will be to use this as an introduction to studying equations over the free group. A more geometrically inclined student could push this project in the direction of algebraic geometry over groups, and a student who leans towards computer science and is already a capable coder could feasibly attempt to reduce the bound of 10,000, possibly all the way down to 2 (I hope!), using linear programming and branch and bound techniques a la the proof of the Kepler conjecture.

  Pre-requisites: Topology and Groups (MATH0074). Recommended: Geometry and Groups (MATH0052).

- **Dr Jonathan Marshall** j.marshall@ucl.ac.uk

  1. *Investigating "secondary" Schottky groups*

     The theory of Schottky groups and automorphic functions, allied with conformal mapping, can be applied to construct explicit solutions to certain classes of problems set in planar multiply-connected domains, many of which are of physical interest. Recently, a special class of these groups - referred to as "secondary" Schottky groups – has been identified. The aim of this project is to investigate these groups and their possible uses in problems of applied mathematics.

     Prerequisites: Complex analysis (MATH0013), and familiarity with Matlab.

- **Prof. Robb McDonald** n.r.mcdonald@ucl.ac.uk

  1. *Interface growth in two dimensions*

     The nonlinear dynamics when an interface deforms in response to a quantity diffusing toward it generates remarkable patterns e.g. viscous fingering, branching stream networks and fractal-like structures formed in electro-deposition. This project will use complex analysis and simple numerical models to explore related models, such as Loewner growth, diffusion-limited aggregation, needle models, and the connections between them.

Pre-requisites: Fluid mechanics (MATH0015) and Complex analysis (MATH0013). Willingness to use and adapt existing numerical models.

2. *Vortex dynamics*

   Investigate and develop analytical and numerical constructions of equilibria for the 2D Euler equations having non-zero vorticity distributions in the form of points, sheets and patches.

   Pre-requisites: Knowledge and enthusiasm for Fluid mechanics (MATH0015 and Complex analysis (MATH0013) is essential.

- **Dr Yusra Naqvi** y.naqvi@ucl.ac.uk

  Positivity properties of symmetric polynomials

  Symmetric polynomials play an important role in the representation theory of symmetric groups and Lie groups. In this project, we would explore a specific family of polynomials (such as Schubert polynomials or Macdonald polynomials) and explain observed positivity of their coefficients relative to a fixed basis by describing them through combinatorial means. This would build on popular models in combinatorics such as Young tableaux, crystals bases and folded alcove walks.

  Prerequisite: Algebra 4 (MATH0053). Recommended: Representation Theory (MATH0073).

- **Prof. Nick Ovenden** n.ovenden@ucl.ac.uk

  1. *Modelling Acoustic Droplet Vaporisation*

     Acoustic droplet vaporisation is a process where an ultrasound pressure pulse induces a phase transition of a liquid nanodroplet into a larger gas bubble. These nanodroplets have applications in terms of embolic occlusion therapy, drug delivery and high intensity focussed ultrasound. Their uptake in clinical practice, however, requires much better understanding of the behaviour of suspensions of nanodroplets in vivo. The mechanics of the acoustic droplet vaporization can be modelling via nonlinear systems of ODEs and PDEs. Possible projects include (i) modelling the vaporization process of nanodroplets, (ii) exploring bubble-nanodroplet interactions or (iii) investigating vascular occlusion with nanodroplets.

  2. *Physiological modelling of critically-ill patients*

     This project will involve working in the mathematics in healthcare hub CHIMERA at UCL looking at biomechanical models of critically-ill patients in intensive care. The project is likely to involve collaboration with clinician. Systems of equations to replicate the respiratory and/or cardiovascular systems will be explored and validated against real-patient data. The project may also incorporate data science/machine learning techniques.

- **Prof. Karen Page** karen.page@ucl.ac.uk

  Mathematical models of diatom ecology

  Diatoms are phytoplankton with beautiful glassy shells (see https://diatoms.org/what-are-diatoms for more details). The student will review species interactions between diatom species and their relevant predators (e.g. herbivorous copepods [1]), and study diatom spatial distributions and movement. They will build ecological models of a selected species of diatom, studying spatial distributions and species interactions. They may also apply species diversity measures.

[1] Pohnert, G., 2005. Diatom/copepod interactions in plankton: the indirect chemical defense of unicellular algae. ChemBioChem, 6(6), pp.946-959.

Prerequisites: MATH0030, programming experience, differential equations. Knowledge of fluid mechanics, especially MATH0024, is also advantageous.

- **Prof. Leonid Parnovski** l.parnovski@ucl.ac.uk

  1. *Periodic operators and lattice points counting*
  2. *Variational approach to spectral theory*

Pre-requisites: Functional Analysis (MATH0018), Multivariable Analysis (MATH0019). Concurrent enrolment in Spectral Theory (MATH0071) would also be desirable.

- **Dr Philip Pearce** philip.pearce@ucl.ac.uk

  1. *Modelling blood flow in vascular networks*
     An intricate network of vessels transports blood between the heart and the rest of the organs in the human body. This project will begin with a review of theoretical models for blood flow in single idealised tubes and in networks of small blood vessels called capillaries. The aim will be to write code in e.g. Python or Matlab to simulate blood flow in various network topologies, and if possible to test how different assumptions about blood properties can be incorporated into such models.
     Pre-requisites: some programming experience; Real Fluids (a co-requisite).
  2. *Multi-scale modelling of living matter*
     The properties and dynamics of biological tissues, organisms and populations emerge from physical and chemical interactions at the levels of molecules and cells. This project can focus on any of these length scales, and can involve a computational or analytical approach. Example projects include: simulating interactions between extracellular matrix proteins in bacterial biofilms; simulating cell populations at the cellular level; or modelling bacterial populations or tissues using a continuum approach.
     Pre-requisites: Some programming experience; Mathematical Methods 4

- **Prof. Yiannis Petridis** i.petridis@ucl.ac.uk

  1. *Lattice counting problems in Euclidean and hyperbolic spaces*
  2. *Ergodic theory of continued fractions*
  3. *The Erdos-Kac theorem on the number of distinct prime factors of the natural number n*
  4. *Selberg's theorem on the normal distribution of the Riemann-zeta function on its critical line*
  5. *L-functions of elliptic curves and effective bounds on class numbers of quadratic fields*

Pre-requisites: Projects normally require Prime Numbers and their Distribution (MATH0083). Depending on the project, Elliptic Curves (MATH0036) or Geometry and Groups (MATH0052) may be useful. For 2. Functional Analysis (MATH0018) is useful.

- **Dr Ian Petrow** i.petrow@ucl.ac.uk

  1. *100% of Galois groups over Q are S_n.*

     When we study Galois theory, we learn to compute the Galois group of a polynomial, or more generally a finite extension of fields. The Galois group of a degree n irreducible polynomial is always a subgroup of the symmetric group S_n. It is natural to ask 'Which subgroups of S_n occur as Galois groups?', and if you just start to write down examples by picking a polynomial 'at random', you will find that you very often get the whole of S_n as its Galois group. In this project, you will make that idea precise and show that, if one orders polynomials of fixed degree with integer coefficients by the maximum absolute value of their coefficients, then, as size of the coefficients gets large, the proportion of polynomials with Galois group the full S_n approaches 100%. The proof of this fact is a beautiful mix of algebraic number theory, analysis (the large sieve), group theory, and prime number theory.

     Prerequisites: Required: Galois Theory (MATH0022) and Number Theory (MATH0022).

     Strongly recommended: Algebraic Number Theory (MATH0035) and Prime Numbers and their distribution (MATH0083).

  2. *Representation of integers by quadratic forms and modular forms*

     Let $Q(x)$ be a positive-definite quadratic form with integer coefficients in at least 3 variables. A basic and old question for each $n > 0$ is how many integral representations of n by the quadratic form $Q(x)$ are there? For some highly structured specific choices of Q, we can give an exact formula for the number of x in $\mathbb{Z}^r$ such that $n = Q(x)$, but in general an exact formula isn't possible. Instead, we look for approximate formulas for the number of solutions as $n \to \infty$. For quadratic forms in 3 or 4 variables the proof of such a formula uses modular forms, which are certain special functions on hyperbolic spaces which have deep connections to modern number theory. The goal of this project is to learn the proof of the asymptotic formula for the representation number and use this as motivation to learn the theory of modular forms. Of particular interest will be certain examples called theta functions, and their role in the proof of the representation number theorem.

     Prerequisites: MATH0051 Analysis 4: Real Analysis MATH0052 Geometry and Groups MATH0083 Prime Numbers and their Distribution

- **Dr Alexey Pokrovskiy** a.pokrovskiy@ucl.ac.uk

  1. *Cycles in graphs*

     Dirac's Theorem is a classic result in graph theory and says that in any graph where each vertex is connected to more than half the other vertices, there is a cycle passing through all the vertics. This project is about looking at various cousins of this theorem e.g. understanding pancyclic graphs - graphs in which there are cycles of all possible lengths.

  2. *Sublinear expanders*

     Recently a tool called"sublinear expanders" has been used to prove a variety of old conjectures from graph theory. Roughly speaking, the tool says that all graphs with a lot of edges contain a large"pseudorandom" chunk. This project will be about understanding the proof of this result and also some applications e.g. to the conjecture of Thomassen about finding cylinders in graphs.

  3. *Other problems about extremal combinatorics and graph theory*

Other projects similar to the above are possible, for example on the topics of “twin-width of graphs” or “hat games on graphs”.

These projects are all suitable to students who have taken Graph Theory and Combinatorics MATH0029 or Combinatorial Optimization MATH0028.

- **Dr Ruth Reynolds** ruth.reynolds@ucl.ac.uk

  1. *Topics in Noncommutative Algebra*

     The following are a suggestion of possible project directions, but I am happy to discuss projects more tailored to the student's specific interests:

     – Noncommutative fields of fractions and Goldie's Theorem;
     – The noetherianity of skew polynomial rings;
     – AS regular algebras and twisted homogeneous coordinate rings;
     – Weyl algebras.

     Suggested prerequisites: Algebra 4, Commutative algebra.

     References:

     Goodearl and Warfield, An introduction to noncommutative Noetherian rings.

     D. Rogalski, An Introduction to Noncommutative Projective Geometry.

     S.C. Coutinho, A Primer of Algebraic D-modules.

- **Dr Mark Roberts** m.l.roberts@ucl.ac.uk

  Projects in non-commutative ring theory. Possible topics include:

  1. *Unique factorisation domains*

     You have seen examples of commutative unique factorisation domains (UFDs), starting in Algebra 2 with Z and a few examples of algebraic rings of integers such as the Gaussian integers $Z[i]$, continuing in Algebra 3 with the polynomial ring $k[x]$. There are more examples in MATH0021 Commutative Algebra and MATH0035 Algebraic Number Theory. There are various ways of generalising the idea of unique factorisation to non-commutative rings and this project looks at the method described in Cohn, *Free Rings and Their Relations*. (This is a very tough read, so don't get put off if you have a look at it!) There is a quite interesting connection with embedding semigroups in groups and rings in fields (see second project) and one could look at the question: if R is a (non-commutative) UFD, does R^*, the semigroup of non-zero elements of R under multiplication, embed in a group?

  2. *Skew fields of fractions*

     This also generalises an idea familiar from commutative algebra: one can embed Z in its field of fractions Q and $k[t]$ in its field of fractions $k(t)$. More generally any commutative integral domain R there is a unique (up to isomorphism) field Q such that R embeds in Q and Q is generated as a field by the image of R (constructed in almost exactly the way as one creates Q from Z). Perhaps surprisingly, the corresponding results fails for non-commutative integral domains: there exists non-commutative integral domains which do not embed in any (skew) field and also non-commutative IDs which embed in non-isomorphic skew fields. The first example of an ID without a skew field of fractions was given by Mal'cev. This project looks at a way of analysing the embedding of rings in skew fields given in Cohn, *Free Rings and Their Relations* (same comment as above), using ideas of matrix ideals.

The pre-requisites for both projects are Algebra 4 (MATH0053) and at least two further algebra modules e.g. Commutative Algebra (MATH0021), Galois Theory (MATH0022), Algebraic Topology (MATH0023), Algebraic Number Theory, (MATH0035), Representation Theory (MATH0073). Commutative Algebra is particularly relevant.

Both projects involve reasonably abstract algebra but also involve quite a bit of algebraic calculations.

- **Dr Calum Ross** calum.ross@ucl.ac.uk

  Topological Solitons

  Certain non-linear models from physics possess self-stabilising configurations known as solitons. In many cases the stability is related to the topology of the space of finite energy field configurations. More practically, topological solitons are particle like lumps of energy that occur in many areas of physics. Examples include vortices in superconductors and superfluids, monopoles and instantons in gauge theories, and skyrmions as models of nuclear matter. Typically, these models have interesting geometric and topological properties.

  This project will involve exploring some mathematical models of vortices, in particular their relationship to rational maps and constant curvature Riemann surfaces, and understanding recent work on integrable vortex equations.

  Prerequisites: MATH0054 (Analytical Dynamics) and MATH0019 (Multivariable Analysis), and Math0020 (Differential Geometry) would be helpful. Good references on the basics of topological solitons include: [1] Manton and Sutcliffe, Topological solitons, Cambridge University Press, 2004. [2] Chapter 6 of Coleman. Aspects of Symmetry: Selected Erice Lectures. Cambridge University Press, 1988.

- **Dr Matthew Schrecker** m.schrecker@ucl.ac.uk

  1. *Shock waves*

     In this project, we will explore the mathematical theory of shock waves in gas dynamics. Shock waves are a fundamental phenomenon in gases, occurring in both large and small scale settings, ranging from in the bell of a tuba to a supernova expansion. The mathematics of shocks goes back to the 19th century, but continues to provide fascinating mathematical problems. The initial aim of the project will be to understand and review some of the classical and more recent literature concerning the behaviour of shocks using tools from mathematical analysis.

     Prerequisites: Basic partial differential equations e.g. as in Methods 4 (MATH0056) or Linear PDE (MATH0070). Recommended: Multivariable Analysis (MATH0019)

  2. *Gravitational collapse*

     In 1939, Oppenheimer and Snyder published a short paper in the field of General Relativity detailing a description of the collapse of a star under its own gravity, leading to the formation of what we might now call a black hole. These elementary solutions have been highly influential in the study of relativity ever since, including for the cosmic censorship conjecture of R. Penrose.

The initial goal of this project is to understand Oppenheimer and Snyder's construction of these basic solutions to the Einstein equations.

Pre-requisites: Mathematics for General Relativity (MATH0025) is recommended.

3. *Singularity formation for non-linear wave equations*

   Although wave equations are one of the basic model equations of PDE, their solutions can exhibit a wide range of behaviours. When even very simple nonlinearities are introduced, the solutions can develop singularities in finite time. In this project, the student will study one of these simple singularity formation mechanisms for a non-linear wave equation.

   Prerequisites: Basic partial differential equations e.g. as in Methods 4 (MATH0056) or Linear PDE (MATH0070). Recommended: Multivariable Analysis (MATH0019)

4. *Compensated Compactness*

   The theory of compensated compactness is a tool in mathematical analysis that is used to study the interaction of weak convergence with nonlinearities. It has been of importance in understanding problems in functional analysis, geometric analysis, and mathematical physics. This project would begin with a review of the basic theory of compensated compactness and some of its applications, but could then extend in a number of different directions, depending on the interests of the student.

   Pre-requisites: Functional Analysis (MATH0018)

- **Dr Edward Segal** e.segal@ucl.ac.uk

  Topics in Geometry, Topology and Algebra:

  I'm happy to discuss potential projects in algebraic geometry, differential geometry, algebraic topology or algebra. A couple of examples are below.

  1. *Representations of quiver algebras*

     Pre-requisites: Commutative Algebra (MATH0021), Representation Theory (MATH0073).

  2. *Principal bundles*

     Pre-requisites: Differential Geometry (MATH0020). Helpful: Topology and Groups (MATH0074).

- **Prof. Michael Singer** michael.singer@ucl.ac.uk

  1. *Geometry of Classical and quantum mechanics*

     In this project we shall explore geometric quantization: symplectic geometry is the correct setting for classical mechanics. Geometric quantization is a recipe (though more of an art-form) for constructing the Hilbert spaces of quantum theory starting from a symplectic manifold.

     Pre-requisite: Multivariable calculus (MATH0019), desirable: Differential geometry (MATH0020). Useful: Analytical Dynamics (MATH0054).

  2. *Other projects in differential geometry*

- **Prof. Frank Smith** f.smith@ucl.ac.uk

  The project(s) will be chosen from the following three areas:

1. *Industrial modelling problems such as in the internal and external flows of fluid associated with vehicle movements on land, sea or air*
2. *Biomedical flows such as through branching vessels or flexibly walled vessels*
3. *Modelling related to sports such as for balls, bouncing and vehicle movements*

   Pre-requisites: the projects above are suitable for students who have taken a full range of methods courses, have experience with the theory of fluids and are interested in ap plying mathematics.

- **Prof. Valery Smyshlyaev** v.smyshlyaev@ucl.ac.uk

  1. *High frequency scattering: asymptotic methods and analysis*

     Problems of wave scattering are mathematically boundary value problems for a PDE. Their approximate solutions for high frequencies can be constructed analytically by a multivariable version of WKB method, which is one of asymptotic methods. Such approximations have a clear physical meaning, and tools of analysis are needed for controlling the accuracy of these approximations.

     Pre-requisites: Waves and Wave Scattering (MATH0080) and Analysis 4 (MATH0051);
  2. *Multi-scale problems and homogenisation: asymptotic methods and analysis*

     Nearly everything around us contains multiple scales, i.e. has often invisible microscopically varying physical properties on which their visible macroscopic properties depend. Mathematically, one needs to deal with boundary-value problems for PDEs with microscopically varying coefficients, and then homogenisation becomes the process of deriving approximate PDEs with macroscopic coefficients. One way of doing this is via asymptotic methods with respect to the underlying small parameter, and the resulting approximations often display interesting physical effects. Tools of analysis are needed for controlling the accuracy of such approximations.

     Pre-requisites: Functional Analysis (MATH0018) and Mathematical Methods 4 (MATH0056).

- **Prof. Alex Sobolev** a.sobolev@ucl.ac.uk

  1. *Pseudo-differential operators*

     Pseudo-differential operators (PDO's) are generalisations of the familiar differential operators. Theory of PDO's forms a tremendously important part of modern Analysis. PDO's are used in Differential Equations, Mathematical Physics, Differential Geometry and many other domains. The aim of the project is to understand the basics of the PDO theory starting with the Fourier transform, PDO calculus and ending with the conditions that guarantee the boundedness of PDO's as linear operators.

     Prerequisites: Analysis 4 (MATH0051), Functional Analysis (MATH0018), Measure Theory (MATH0017) is helpful but not critical.
  2. *Mathematical theory of wavelets*

     The goal of the theory is to find a function on the real line such that the set of its translates and its rescaled copies forms a basis of $L^2(\mathbb{R})$, the space of square integrable functions. Such a function is called a wavelet. The aim of the project is to work through the Multiresolution

Analysis which constitutes the basis of the whole approach, and to understand some known examples of wavelets.

Prerequisites: Analysis 4 (MATH0051), Functional Analysis (MATH0018).

- **Prof. Alan Sokal** a.sokal@ucl.ac.uk

  Continued fractions in enumerative combinatorics

  Continued fractions arise in number theory as representations of irrational numbers. Here we are concerned, by contrast, with continued-fraction representations of (possibly divergent) power series. For instance, Euler proved in 1746 that

$$\sum_{n=0}^{\infty} n!\, t^n \;=\; \cfrac{1}{1 - \cfrac{1t}{1 - \cfrac{1t}{1 - \cfrac{2t}{1 - \cfrac{2t}{1 - \cfrac{3t}{1 - \cfrac{3t}{1 - \cdots}}}}}}} \,.$$

  This identity makes sense as a *formal power series*, even though the left-hand side is divergent for all $t \neq 0$. Since $n!$ counts the permutations of an n-element set, it is natural to want to refine this continued fraction by counting permutations with respect to some combinatorially interesting statistics (for instance, the number of cycles). Many such formulae have been found by enumerative combinatorialists in the last few decades. This project would involve learning about this work, and possibly finding new examples by computational exploration.

  Prerequisites: Complex Analysis (MATH0013) and the elementary theory of polynomial and formal-power-series rings (MATH0014) all come in. Some knowledge of combinatorics (e.g. MATH0029) would also be useful. A good background reference in enumerative combinatorics is Wilf's *Generatingfunctionology*. Finally, a good knowledge of Mathematica will be essential to the computational side of this project.

- **Dr David Solomon** d.r.solomon@ucl.ac.uk

  1. *p-adic Numbers, p-adic Analysis and Applications to Number Theory*

     This project lies intriguingly at the interface of Analysis and Number Theory. The 'p' in the title is a prime number and to each such p there corresponds a topologically complete field, the p-adic numbers. It is a bit like the field of real numbers but with some very striking – and, initially, rather disorienting - differences. In particular, it is 'non-Archimedean'. In this project, we will first construct the p-adic numbers then do some elementary p-adic analysis with them including the construction of p-adic logarithmic, exponential and Gamma functions. These turn out to be far easier to handle than the real and complex functions of which they are analogues, which is fortunate when we apply the Analysis back to the study of Number Theory. One application uses the p-adic Gamma-function to prove 'supercongruences' with respect to p for binomial coefficients. Another, more ambitious, uses p-adic L-functions to study the p-divisibility of the class-numbers of certain number fields.

Pre-requisites: MATH0034. Also, MATH0035, depending on applications chosen. Plus a solid understanding of standard 1st and 2nd-year Real and Complex Analysis.

2. *Binary Quadratic Forms, Prime Numbers and Class-Groups*

   An integral binary quadratic form (IBQF) is a homogeneous quadratic expression in two variables with whole-number coefficients. Today, we view 'class-groups', 'units' etc. as lying in the domain of algebraic number fields but historically speaking, their origins lie in the study of IBQFs by the great Carl Friedrich Gauss (1777-1855) as well as other 18th and 19th-century mathematicians. Today, IBQFs and their generalisations remain relevant to both computational and theoretical number theory. This project will start by examining the basic notions of IBQFs: definiteness, representability and equivalence. We will look at the very classical question of which prime numbers are values of such forms (generalising the famous two-squares theorem) which leads us to a first glimpse of class-field theory. We will also study the application of IBQFs to the computation of class-groups for quadratic number fields. Pre-requisites: MATH0034 and MATH0035.

3. *Cyclotomic and Abelian Fields with Applications*

   A cyclotomic field is a number field generated over the rationals by a root of unity. Such fields form one of the most intensively studied classes of number fields because of their explicit nature and the fact that any abelian field - a Galois extension of the rationals with abelian Galois group - is contained in a cyclotomic field. (This is the Kronecker-Weber Theorem.) First, we will analyse in more detail the objects from MATH0035 in the specific case of cyclotomic fields: rings of integers, class groups and units but also objects peculiar to cyclotomic fields: Jacobi Sums, Gauss Sums, and Cyclotomic units, for instance. Emphasis will be placed on their structure as modules for the Galois group. We will then apply the theory to one or more of the following: Stickelberger's Theorem, Thaine's Theorem, proof of the Catalan Conjecture, certain pre-Wiles cases of FLT and the Kronecker-Weber Theorem.

   Pre-requisites: MATH0034, MATH0035 and MATH0053. Some ideas from MATH0073 would also useful.

- **Dr Isidoros Strouthos** i.strouthos@ucl.ac.uk

  1. *Topics involving topology and/or abstract algebra (possibly also including some applications related to other areas, e.g. in physics and/or biology)*

     Even though different relevant projects might correspond to different (collections of) relevant pre-requisite modules, such projects may involve material covered in modules such as 'Algebra 3: Further Linear Algebra' and 'Algebra 4: Groups and Rings' (as well as, perhaps, some material involved in one or more of the modules 'Algebraic Topology', 'Commutative Algebra', 'Representation Theory'); please feel free to contact me directly for more information regarding possible relevant projects."

- **Dr Sergei Timoshin** s.timoshin@ucl.ac.uk

  1. *Two-fluid flows*

Two-fluid flows can be studied in various approximations which reflect the specifics of the flow (e.g. thin layers), in two and three dimensions, with or without explicit time dependence. There are many interesting and unsolved problems related, for example, to flow separation and instability.

Prerequisites: Knowledge of fluid dynamics at the level of Real Fluids (MATH0077) is essential.

- **Dr Matthew Towers** m.towers@ucl.ac.uk

  1. *Universal enveloping algebras*

     There is a ring U, called the 'universal enveloping algebra', associated to each Lie algebra L. The project will study the universal enveloping algebras associated to the special linear Lie algebra sln for $n = 2$ or 3 over fields of characteristic 2. The specifics will depend on the interests and the background of the student undertaking the project: for example, we might look at the centres of these algebras, their algebras of derivations, or their cohomology.

     Pre-requisites: Algebra 4 (MATH0053), and some of Commutative Algebra (MATH0021), Algebraic Topology (MATH0023), Representation Theory (MATH0073). Recommended concurrently: Lie groups and Lie algebras (MATH0075).

  2. *Category theory and topology in functional programming*

     One possible project begins by giving an introduction to category theory in the context of functional programming. After that it would investigate semantic approximation order, recursive definitions as fixed points, or monads and their algebras.

     Another idea would be to give an account of some remarkable work of Martin Escardo `https://www.cs.bham.ac.uk/~mhe/.talks/popl2012/escardo-popl2012.pdf` on decidable equality for function types.

     Pre-requisites: some programming skills, and knowledge of or willingness to learn a functional language, e.g. Haskell, Scheme.

  3. *The mathematics of Enigma*

     This project is about the mathematics of Enigma, the World War 2 cipher used by the German armed forces which was broken first by work of Polish mathematicians and then by the British at Bletchley Park. Many variants of Enigma were used and many different techniques used to decipher it using the very limited computational power available at the time. The project would study and explain some of these methods from a modern point of view. No formal prerequisites, but a basic knowledge of statistics and willingness to do some programming and a lot of reading will help.

- **Prof. Jean-Marc Vanden-Broeck** j.vanden-broeck@ucl.ac.uk

  1. *Analytical and numerical studies of waves of large amplitude*

     Pre-requisites: Fluid Mechanics (MATH0015) or equivalent.

- **Dr Benjamin Walker** benjamin.walker@ucl.ac.uk

1. *Models of tumour spheroid growth* Mathematical models of tumour growth have been around for decades, leading to a range of different modelling approaches. This project will consider simple models of tumour development and compare them, exploring how the choice of model can impact the conclusions that we might draw. In particular, the project will look at how the effects of treatment can be incorporated into the various models and whether we can meaningfully translate parameters between them. The final goal will be to (numerically) assess how important the choice of model is in optimising dosage scheduling, a key question in modern healthcare.

   Prerequisites: Familiarity with MATH0030 (Mathematical Ecology) and non-linear ODEs (as taught in MATH0027) is desirable. Experience with programming in Python or MATLAB is helpful but necessary.

2. *Measuring microswimming* Small-scale swimmers often move by beating a long, slender tail in a sinusoid-like shape. However, the details of this beat, such as its amplitude or frequency, may vary over time. This project will explore and test new ways to define, measure, and analyse the properties of the beating tail as they change over time. This will be tested against synthetic data and then applied to the results of real-world imaging from a canonical microswimmer. Despite the biological application, no familiarity with fluid mechanics is required.

   Prerequisites: Experience with programming in Python or MATLAB is recommended.

3. *Asymptotics of oscillatory swimming* Many swimmers oscillate rapidly as they swim, leading to trajectories that look smooth over long timescales but are intricate and complex over short timescales. This project will combine a multiple-scales asymptotic analysis of simple ODE models with numerical simulation, looking to determine the relationship between small-scale oscillations and large-scale behaviours of self-propelled particles, potentially including the effects of fluid flow.

   Prerequisites: Some familiarity with asymptotics, such as MATH0078 and the method of multiple scales, is needed. Experience with programming in Python or MATLAB is helpful but necessary.

- **Prof. Alexey Zaikin** alexey.zaikin@ucl.ac.uk

  1. *Modelling complex behaviour of organ-on-a-chip platforms with linked mechanosensitive and genetic dynamics*

     Organ-on-a-chip platforms have the potential to accurately predict human physiology and, especially, diseases. The idea of the project is to develop for the first time a mathematical model of a such a system with linked mechanosensitive viscoelastic properties and complex dynamics of intracellular genetic networks. Recently, we have shown that genetic networks may have very complex dynamics [1-3]. On the other hand, recent studies suggest bidirectional causal links between cellular clocks and mechanotransduction [4]. The modelling can be interesting for investigation of organ-on-a-chip systems, which aim to mimic and predict organ-level human physiology by incorporating 3D co-culture of multiple cell types and physiologically relevant mechanical stimuli to recapitulate the in vivo cellular environment [5].

     The project is of a computational nature and will require numerical simulations and solutions of a system of coupled ordinary differential equations.

     [1]. L. Abrego, and A. Zaikin, "Integrated Information as a Measure of Cognitive Processes in Coupled Genetic Repressilators", Entropy 21(4), 382 (2019). [2]. R. Bates, O. Blyuss, and A. Zaikin," Stochastic resonance in an intracellular genetic perceptron", Phys. Rev. E, 89,

032716 (2014). [3]. Y. Borg, E. Ullner, A. Alagha, A. Alsaedi, D. Nesbeth, and A. Zaikin, "Complex and Unexpected dynamics in Simple Genetic Regulatory Networks", IJMPB 28, 1430006 (2014). [4]. Yang, N et al. Nat Commun (2017). DOI: 10.1038/ncomms14287 [5]. Thompson, CL et al. Front Bioeng Biotechnol (2020).