Project Titles for MATH0084
2021-22

Suggested projects for 2021-2022 are arranged here alphabetically by supervisor. Most projects have either prerequisite or suggested modules. Please contact supervisors directly if you would like to find out more information about a project or to register an interest in taking that project.

- **Dr Stephen Baigent** steve.baigent@ucl.ac.uk

1. *Invariant manifolds of discrete-time dynamical systems*

   Many models that arise in theoretical ecology and evolutionary game theory have curves, surfaces, or more generally manifolds, that are left invariant by the dynamics. For models that have an invariant manifold that attracts all points, the model can be solved by restricting to the manifold, generally an easier problem. The first aim of the project will be to learn some key mathematical theory for showing when these invariant manifolds exist, and then to apply the theory to several well-known models from theoretical ecology. A second aim will be to find (hopefully new) ways of computing these invariant manifolds using a program such as Mathematica, Python or Matlab, and then use the computations to push the models to limits where the invariant manifolds lose smoothness and eventually disappear.

   Pre-requisites: Some experience of programming would be useful. No Mathematical Biology background is needed.

2. *Optimal habitat choice with travel costs*

   This project builds on one supervised 2 years ago. Imagine a large population consisting of several species that populate a fixed habitat. Each habitat has a range of resources and each species has food preferences, safety concerns, etc. How does the population fill out the habitat? This is an evolutionary game and the choice is usually a special kind of Nash equilibrium known as an evolutionarily stable state. If it actually costs individuals to move between sites, this complicates the problem: They may not move even if they would be 'fitter' in the new site if it costs too much in fitness terms to reach it. The theory for this is not so well known, and possibly not known for some models. The aim will be to formulate a new 2D partial differential equation model where there is not a finite number, but a continuum of species. The first task will be to set up the model and show that it makes sense (has meaningful solutions for reasonable scenarios) and the second task will be compute these solutions using finite-differences, finite elements, or similar, using Mathematica, Python, Matlab or a suitable pde solver, and explore cost-benefit trade-offs for different models.

   Pre-requisites: Some experience of programming would be useful. While the project includes some game theory ideas covered in the 2nd term module Evolutionary Games and Population Genetics (MATH0082), the project can be done independently.

- **Dr Costante Bellettini** c.bellettini@ucl.ac.uk

1. *Minimality of the Simons cone*

Discovered to be a stable minimal hypersurface by J. Simons, this 7-dimensional cone in $\mathbb{R}^8$ was proved to be a minimizer of the area by Bombieri, De Giorgi and Giusti (about 50 years ago). The discovery of this area-minimizer with a singular point indicates that the minimizing problem has to be posed in a class of non necessarily smooth hypersurfaces (sets of finite perimeter in geometric measure theory, a field where measure theory and differential geometry merge). A singularity formation of this type does not arise in dimensions up to 6 and it is still mysterious nowadays what makes dimension 7 so special.

2. *Allen-Cahn energy and minimal surfaces*
   The Allen-Cahn equation is a second order elliptic semi-linear partial differential equation used to describe phase separation of a two-phase liquid (two components of a binary fluid spontaneously separate and form domains that are pure in each component). In recent years this PDE has had striking impact on geometric problems and active research is ongoing in this direction. The connection with geometry lies in the fact that the interface of separation between the two phases of the liquid tends to be locally area minimizing: the liquid tries to use as little area as possible to transition from one phase to the other. This feature of the Allen-Cahn energy has prompted, for example, a new and rather straightforward proof of the following fundamental result (originally proved in the 80s): in every closed Riemannian manifold there exists a closed minimal hypersurface. Minimal means that the mean curvature is everywhere zero - this is the geometric counterpart of the area minimization property. The lowest-dimensional example of a minimal hypersurface is a geodesic on a surface (e.g. an equator on a sphere).
3. *Other topics in geometric measure theory and in elliptic partial differential equations (Monge-Ampere equation, harmonic maps, etc).*

Pre-requisites for these projects: Analysis 4 (MATH0051). Any of Measure Theory (MATH0017), Linear Partial Differential Equations (MATH0070) and Differential Geometry (MATH0020) would be helpful.

- **Prof. Timo Betcke** t.betcke@ucl.ac.uk
  1. *Representation of electromagnetic fields through fundamental solutions*
     In many practical algorithms we need to numerically represent electromagnetic solutions through simple basis functions. In this project we want to focus on the representation of solutions of Maxwell equations through the use of fundamental solutions. Interesting questions here are convergence properties and low-rank approximations to compress field representations. This project involves significant programming and good Python knowledge is expected.

- **Dr Christopher Birkbeck** c.birkbeck@ucl.ac.uk
  1. *p-adic rings in number theory*
     p-adic numbers are widespread throughout number theory, algebra and geometry. One of the reasons they are so useful is that the ultrametric inequality gives them a particular topology which plays well with algebraic objects. This project has two directions in which one can go in: one is studying certain p-adic rings, their properties and some uses and the other is studying p-adic matrices:

- Perfectoid rings: The aim of this project is to study more advanced versions of p-adic rings which underlie many of the most important recent developments in geometry and number theory. Most notably is work by Scholze which heavily uses perfectoid rings. The project would begin by first learning some p-adic field theory and from there basic examples of perfectoid fields. From there it would move to general perfectoid rings and their properties. (This project is essentially lots of advanced ring/field theory).
- p-adic matrix analysis: Many of the standard results regarding matrices over reals and complex numbers help us understand eigenvalues and singular values of matrices. In the case of matrices with p-adic entries, most of these results have analogues but the resulting structures have not been widely studied, for example their Smith normal forms. The goal of this project would be to generalise some well-known results to the p-adic setting and study certain conjectures regarding their Smith normal forms. Some experience with (or willingness to learn) computational tools such as Sage/Mathematica/Magma would be useful for studying these conjectures. (This project is essentially advanced linear algebra with some computational aspects).

Pre-requisites: MATH0034 and at least two of MATH0035, MATH0022, MATH0021, MATH0051.

- **Dr Christian Boehmer** c.boehmer@ucl.ac.uk

1. *Continuum mechanics with microrotations*

   The aim of this project is to study elasticity theory in the presence of micro-rotations. This theory is known under a few different names like Cosserat elasticity or Micropolar elasticity. As a first step the candidate would have to become familiar with elasticity theory (linear and non-linear) and next include micro-rotations. Various routes could be explored ranging from more computational work using Mathematica or more analytical work which would involve the calculus of variations to study equations of motion.

   Pre-requisites: No particular prerequisites.

2. *Modified theories of gravity with diffeomorphism non-invariance*

   The first part of the project is to study the variational approach to the Einstein field equations and looking at the original Einstein action, sometimes called the Gamma squared action, which is different from the Einstein-Hilbert action commonly used. This can be used to set up a modified theory of gravity with second order field equations similar to those found in other popular modified gravity models. Interestingly, this model is no longer diffeomorphism invariant in general. The main part of the project is about studying this model in some concrete situations like cosmology, spherical symmetry or the study of gravitational waves. There are many avenues that can be explored further.

   Pre-requisites: Mathematics for General Relativity (MATH0025)

3. *Equations of motion in General Relativity*

   General Relativity is a unique theory in the sense that the field equations imply equations of motion. This is in stark contrast to Electromagnetism, for instance, where the Lorentz force has to be imposed and is not part of the field equations. This project looks at the equations of motion of a small charged particle and treats this as a perturbation of flat spacetime. By carefully studying the perturbed Einstein field equations, one can derive the Lorentz force explicitly from the theory.

   Pre-requisites: Mathematics for General Relativity (MATH0025) essential.

Pre-requisites: Please note that most of these projects require a good deal of programming in Mathematica. It is therefore essential that candidates have some programming background and are willing to invest effort into learning Mathematica.

- **Dr Robert Bowles** r.bowles@ucl.ac.uk

  1. *The impact of viscous effects on free surface flow*

     The project examines a range of viscosity-influenced free-surface flows, for example planar and axisymmetric flows, flow over obstacles, hydraulic jumps. Details can be discussed with the student. Some computational work is likely to be required.

     Prerequisites or to take concurrently: Real Fluids (MATH0077), Asymptotic Approximation Methods (MATH0078)

  2. *The flow emerging from a tap*

     This project analyses the flow of a Newtonian fluid as it emerges from a vertical circular pipe into the open air. Important physical effects such as the switch from no-slip to no-shear boundary conditions, the impact of surface tension and the inertia of the jet itself can be analysed and their interactions studied. Extensions to non-Newtonian fluids may be possible. Computational work is required.

     Prerequisites (co-requisite): Real Fluids (MATH0077), Asymptotic Approximation Methods

- **Prof. Erik Burman** e.burman@ucl.ac.uk

  1. *Inverse boundary value problem for elastodynamics*

     In this project we will consider the equations of elastodynamics and explore numerically how the solution can be reconstructed when the data on part of the domain boundary is unavailable. To compensate for this lack of data we assume that some additional measurements are available, either in the bulk domain or on the boundary. The package FreeFEM++ will be used for the numerical computations.

     Pre-requisites: Variational Methods for PDEs (MATH0092)

  2. *Nonlinear viscosities for conservation laws*

     We will consider the Burgers equation in one space dimension and study how the addition of artificial viscosity can improve the behaviour of computational methods. In particular we will compare the effect of linear and nonlinear diffusivities close to and far away from discontinuities in the solutions

     Pre-requisites: Numerical Methods

- **Dr Shane Cooper** s.cooper@ucl.ac.uk

  1. *Mathematical approach to innovative composite material design*

     In this project we shall study solutions of second-order partial differential equations with rapidly oscillating coefficients using asymptotic analysis. The equations of interest arise from mathematical models for the behaviour of modern advanced man-made composite materials.

- **Dr Matthew Crowe** m.crowe@ucl.ac.uk

  1. *Geostrophic turbulence*
     On the large scale, the ocean and atmosphere can be modelled as thin fluid layers hence motion is predominantly in the (2D) horizontal plane. Of particular interest is turbulent motion which results in the formation of large geostrophic eddies. The first aim of this project is to review the theory of geostrophic turbulence using both analytical and numerical approaches. The second aim will be to examine how strong vertical mixing modifies the structure of geostrophic turbulence. This part is expected to predominantly use numerical simulations though it may also be possible to make some progress using a modification of classical 2D turbulence theory.
     Pre-requisites: Methods 4 (MATH0056), Fluid Mechanics (MATH0015) and some knowledge of Python will be useful.
  2. *Solutions of Monge-Ampere equations*
     Monge-Amprere equations are a type of nonlinear PDE which commonly arise in oceanic and atmospheric problems when considering the vorticity of the system. The nonlinear nature of these equations makes finding numerical solutions difficult. The aim of this project is to find a way of transforming these equations to a linear form which can be solved numerically and to implement the transformation and solution. The transformation may give some insight into when solutions exist. This project is open ended with scope for a more 'theoretical' Analysis-based approach or a more 'practical' numerical approach.
     Pre-requisites: These vary with the project direction. Any of; Methods 4 (MATH0056), Fluid Mechanics (MATH0015), Analysis 4 (MATH0051), Multivariable Analysis (MATH0019) and Linear PDE (MATH0070) may be useful.
  3. *Internal waves in a sloping channel*
     Internal waves occur in fluids where the density changes with depth and are common features of the ocean and atmosphere. When analytically examining waves in a channel, we often assume the walls are vertical allowing simple boundary conditions to be applied. The aim of this project is to use an asymptotic approach to consider the effects of sloping side-walls and bottom and to derive a single (KdV-like) equation to describe the evolution of the wave amplitude. This project is primarily analytical though there is scope for numerical simulations if desired.
     Pre-requisites: Fluid Mechanics (MATH0015). Geophysical Fluid Dynamics (MATH0024) would be useful.

- **Dr Mohit Dalwadi** Mohit.Dalwadi@maths.ox.ac.uk

  1. *Fundamental models of multiscale mass and fluid transport*
     Multiscale problems of mass transport are ubiquitous in physical applied mathematics. Applications include fluid transport in tumours, membrane filtration, nutrient delivery to plant roots in soil, salt transport in sea ice formation, and many more. In this project, the student will review basic partial differential equation models for multiscale mass and fluid transport, and go on to investigate asymptotic solution structures when regions involving different dominant transport mechanisms are coupled together. There are also opportunities - but no requirements - to write numerical simulations in this project.
     Pre-requisites: Advanced Modelling Mathematical Techniques (co-requisite), Asymptotic Approximation Methods (co-requisite), Mathematical Methods 5 [or a willingness to learn aspects of each].

2. *Cryopreservation*

   Cryopreservation technology is used for applications involving fertility, tissue transplantation, and the protection of endangered species. Mathematical models can be used to understand how to reduce cell damage in this process. Since freezing and melting involve transitions between ice and water phases, mathematical models of this process can involve solving partial differential equations with moving boundaries, where the position of the domain boundary must be determined as part of the solution. In this project, the student will review basic mathematical models for freezing, then investigate how adding cryoprotective chemicals can reduce cell damage in cryopreservation. There will be opportunities to use both asymptotic and numerical methods in this project.

   Pre-requisites: Advanced Modelling Mathematical Techniques (co-requisite), Asymptotic Approximation Methods (co-requisite), Mathematical Methods 5 [or a willingness to learn aspects of each].

3. *Decontaminating chemical agents*

   The use of the chemical warfare agent Novichok in the Salisbury poisonings demonstrates the importance of being able to thoroughly decontaminate affected areas. Mathematical models can be used to understand how to choose appropriate cleansers when confronted with novel chemical agents in the field. Such models typically involve solving partial differential equations with moving boundaries, where the position of the domain boundary must be determined as part of the solution. In this project, the student will review basic mathematical models for chemical decontamination, then explore more complex set-ups, such as emulsions of agent and cleanser. There will be opportunities to use both asymptotic and numerical methods in this project.

   Pre-requisites: Advanced Modelling Mathematical Techniques (co-requisite), Asymptotic Approximation Methods (co-requisite), Mathematical Methods 5 [or a willingness to learn aspects of each].

- **Dr Ben Davies** ben.m.j.davies@ucl.ac.uk

  1. *Automating assessments for tertiary mathematics in STACK*

     STACK is a new programming language developed to aid learning designers in the automation of assessment in mathematics. This tool, housed in Moodle and built on Maxima, allows designers to randomise question variants, personalise feedback based on student input, and develop intricate pedagogic sequences based on sophisticated mathematical code. In this project, the student will be expected to produce a suite of mathematically sophisticated question models, and to develop a series of design principles informing future design. An interest in education and mathematical pedagogy is a must, and previous coding experience is highly recommended.

  2. *Proof comprehension and the role of summarising in mathematical argumentation*

     Mathematicians rarely write out a mathematical proof 'in full', with explicit mentions of axioms defining their operational space and without any implicit warrants linking statements. Rather, we rely on shared conventions to abbreviate their mathematical arguments to only the most salient derivations. As such, 'summarising' is central to mathematical practice. In this project, the student will take a deep theoretical look at the activity of summarising across disciplines, and will apply these ideas in the realm of mathematical proof. The project lies at the intersection of mathematical, (experimental) philosophy and statistics, and could follow many different paths according to the student's interest.

3. *Spread of mathematical knowledge*

   The mathematical community shares knowledge via a variety of different media. Despite the innovations of the 21st century, peer-reviewed journals and chalk-and-talk seminars remain pivotal in the spread of mathematical knowledge. In this project, the student will study this spread by taking an active role within the community of research mathematics. The student will be expected to attend a seminar series of their choosing, and to document the experience from the position of a 'participant-observer'. This ethnographic investigation will focus both on the mathematical insights gained, and the ways in which these new insights came about. This project will suit a confident, self-reflective individual, capable of detailed note-taking and deep reflection on their own learning process.

- **Dr Alejandro Diaz** alex.diaz@ucl.ac.uk

  1. *Calibration of computer models through History Matching*

     Computer models are essential in modern science to study the behaviour of complex systems. The reliability of these models depends on how well they are calibrated to data. However, some models are extraordinarily expensive, and a reduction of the input space prior to calibration is needed to mitigate the computational cost. History Matching is a technique designed for this, by carefully sampling the input space, given all the known sources of uncertainty. The aim of this project is to design strategies to make model calibration more efficient, by either establishing new theoretical properties or defining new sampling algorithms. The project requires familiarity with basic probability theory and proficiency in coding. Matlab, Python and R are suitable programming languages.

- **Dr Vladimir Dokchitser** v.dokchitser@ucl.ac.uk

  1. *The inverse Galois problem and elliptic curves*

     It is a long-standing unsolved problem whether every finite group is the Galois group of a polynomial over the field of rational numbers. The aim of the project is to make a survey of some known results on the problem in general, and then describe in detail how to use elliptic curves to construct extensions with certain Galois groups.

     Prerequisites: Galois Theory (MATH0022) and Elliptic Curves (MATH0036) are essential. Algebraic Geometry (MATH0076), Representation Theory (MATH0073) and Algebraic Number Theory (MATH0035) are desirable.

- **Prof. Gavin Esler** j.g.esler@ucl.ac.uk

  1. *Dam breaks in the presence of friction*

     The 'dam break' is a classical similarity solution of the shallow water equations. The aim of this project is to investigate how the situation changes when friction is present, using asymptotic methods. One key aim will be to determine the circumstances in which 'micro-breaking' occurs in the solutions, at the rear of the head of the resulting gravity current.

Pre-requisites: MATH0015 Fluids, MATH0024 Geophysical fluids OR MATH0077 Real fluids. MATH0056 Methods IV or similar methods course.

2. *Is the quasi-biennial oscillation a noise amplifier or damper?*

   One of the strangest phenomena in the Earth's atmosphere is the quasi-biennial oscillation (QBO) of the mean winds in the equatorial stratosphere. The winds evolve from easterly to westerly with a (slightly irregular) period of around 28 months. The oscillation is understood to be due to wave-mean flow interaction, with waves generated at the Earth's surface propagating upwards and depositing momentum in the easterly or westerly jet, driving its evolution. Rather simple mathematical models of the QBO can capture its basic features. The aim of the project is to add a stochastic component to the wave forcing in one of these models, to discover how randomness in the forcing translates into randomness in the QBO itself, and to answer the question posed in the title.

   Pre-requisites: MATH0015 Fluids, MATH0024 Geophysical fluids OR MATH0077 Real fluids. Ideally have encountered stochastic process e.g. in MATH0031 Financial maths, and have taken mainly applied / methods courses.

- **Dr Lorenzo Foscolo** l.foscolo@ucl.ac.uk

1. *Morse theory*

   The project is about understanding an instance of the deep interactions between analysis on manifolds and the geometry and topology of the underlying space. We will consider a manifold M and a smooth function f on M with non-degenerate (i.e. not saddle) critical points; for example, we can let M be a surface in Euclidean space and f a height function. Morse theory is about reconstructing the topological shape of M from the critical points of f.

   Prerequisites: Multivariable Analysis (MATH0019), Differen'tial Geometry (MATH0020); (co-requisite) Algebraic Topology (MATH0023) or Topology and Groups (MATH0074)

2. *The angle criterion*

   The Plateau Problem asks about the surface with minimal surface area amongst all surfaces with a fixed boundary. For example, if you have a circle contained in a plane in Euclidean space, then the disk is the area minimising surface with the given boundary. (All of this makes sense not only for surfaces in 3-space but also n-dimensional submanifolds of Euclidean space of arbitrary dimension.) In this project we want to answer the following question: when is the union of two planes area minimizing? There is a complete answer to this question in terms of the angles between the two planes.

   Prerequisites: Measure Theory (MATH0017), Multivariable Analysis (MATH0019), Differential Geometry (MATH0020).

- **Dr Jeffrey Galkowski** j.galkowski@ucl.ac.uk

1. *Scattering theory*

   In this project, will will explore the mathematics of waves interacting with perturbations of free space e.g. light scattering off of a glass. This mathematics lies behind technologies such as CT scan imaging and is found in the study of topics as far reaching as acoustics and general relativity.

2. *Semiclassical analysis*

   In this project, we will explore the mathematics which links high energy quantum behaviour to classical dynamics. This theory roughly amounts to describing functions and especially solutions to partial differential equations locally in both position and momentum and exploring the effects that the structure of a PDE has on its solutions.

Prerequisites: Basic partial differential equations e.g. as in Methods 4 (MATH0056), Functional analysis (MATH0018), (co-requisite) Spectral theory (MATH0071)

- **Dr Luis Garcia Martinez** luis.martinez@ucl.ac.uk

1. *Heegner points and the congruent number problem*

   A rational number is said to be a congruent number if it is the area of a right triangle with rational sides. The goal of this project is to understand Ye Tian's work on the ancient problem of determining all congruent numbers. His approach uses an explicit construction of rational points on elliptic curves due to Heegner, combined with Kolyvagin's Euler system method.

   Prerequisites: Elliptic Curves (MATH0036). Recommended: Algebraic Number Theory (MATH0035).

2. *Arithmetic of Dynamical Systems*

   A diophantine equation is a polynomial equation that is to be solved in integers. In recent years techniques from the theory of dynamical systems have been used to obtain non-trivial results about diophantine equations. The project would be to understand some of these results, focusing on the relation between dynamical systems and heights.

   Prerequisites: Elliptic Curves (MATH0036). Recommended: Algebraic Number Theory (MATH0035).

- **Dr Luca Grieco** l.grieco@ucl.ac.uk

1. *Operational Research in the management of disasters*

   The aim of this project will be to use Operational Research techniques for the management of healthcare operations in the context of disasters. The choice of the specific area of application (i.e. which type of disaster, which type of problem) will be discussed and agreed with the interested student. Examples include (but are not limited to): allocation of personal protective equipment or other resources to prepare for CBRN (chemical, biological, radiological, nuclear) disasters, resource management in acute hospitals or in home healthcare organisations, long-term COVID-19 vaccination strategies, prediction of healthcare demand in space and time after a disaster. The student will explore the scientific and grey literature about existing approaches regarding the problem of interest, formulate a mathematical model tackling a relevant specific question, implement it using appropriate software, analyse it to assess its usefulness and identify challenges regarding applicability of the approach in real life.

   Pre-requisites: some knowledge of programming would be useful; depending on the approach followed, knowledge of some probability theory would be advantageous.

- **Dr Mahir Hadzic** m.hadzic@ucl.ac.uk

1. *Derivation of the Vlasov-Poisson system*

   At the centre of the project is the Vlasov-Poisson system - a fundamental kinetic equation used in the description of plasmas and galaxies. An outstanding open question in the field is to rigorously derive the Vlasov-Poisson equation as a continuum limit of the discrete N-body problem as N goes to infinity. The goal of the project is to get an overview, understand the existing literature, and focus on recent partial progress on this problem.

   Prerequisites: Analysis 4 (MATH0051), Recommended: Multivariable Analysis (MATH0019).

2. *Wave equations outside obstacles*

   We consider the wave equation outside a compact obstacle. The goal is to understand the decay-in-time properties of the solution assuming suitable boundary conditions on the boundary of the obstacle. We shall consider the Dirichlet, the Neumann, and the Robin boundary conditions. Our starting point is the seminal work of Morawetz from 1960's which relies on the so-called multiplier / vector-field method.

   Prerequisites: Analysis 4 (MATH0051), Recommended: Multivariable Analysis (MATH0019).

- **Prof. Rod Halburd** r.halburd@ucl.ac.uk

1. *Quaternionic/hypercomplex analysis*

   There have been several attempts to create a satisfactory analogue of complex analysis for functions of a single quaternionic variable. We will explore a fairly recent development: the theory of slice-regular functions. Beyond understanding this theory, the aim of this project is to attempt to extend several theorems from complex analysis and/or the theory of special functions to this setting.

   Pre-requisites: None beyond core modules.

2. *Asymptotics beyond all orders*

   Abel once wrote that "Divergent series are the invention of the devil, and it is shameful to base on them any demonstration whatsoever."

   This project will (1) study some divergent series and then (2) base some demonstrations on them (i.e., we'll prove some stuff). More precisely we will use the theory of asymptotic series (and the much more recent theory of asymptotics beyond all orders, also known as exponential asymptotics) to study the solutions of certain differential equations. We can either attempt to obtain rigorous growth estimates of solutions in the complex domain or consider an application in quantum mechanics.

   Pre-requisites: None beyond core modules.

3. *Topics in general relativity*

   Possible directions include (1) the role played by different choices of coordinates and gauge, (2) approximations methods and (3) gravitational waves.

   Pre-requisites: Mathematics for General Relativity (MATH0025) (essential)

- **Dr Ali Hammad** a.hammad@ucl.ac.uk

1. *Project on rheological exploration of complex fluids*

This project uses rheology to explore complex fluids and derive a new equation for isoperimetric inequalities and curve shortening flow on surfaces. The following links provide useful articles for the project:

Gillissen, J; Papadopoulou, A; Balabani, S; Tiwari, M; Wilson, H; (2020) Suspension rheology of adhesive particles at high shear-rates. Physical Review Fluids, 5 (5), 10.1103/PhysRevFluids.5.053302.

Schulze, F; (2020) Optimal isoperimetric inequalities for surfaces in any codimension in Cartan-Hadamard manifolds. Geometric and Functional Analysis, 30 pp. 255-288. 10.1007/s00039-020-00522-8.

Dokchitser, V; Evans, R; Wiersema, H; (2020) On a BSD-type formula for L-values of Artin twists of elliptic curves. Journal für die reine und angewandte Mathematik 10.1515/crelle-2020-0036.

The prerequisites or recommend courses for the project are MATH0019 – Multivariable Analysis; MATH0074 – Topology and Groups.

- **Dr Betti Hartmann** ucahbha@ucl.ac.uk

  1. The student would be working on a project related to nonlinear effects in classical field theory. The resulting coupled differential equations need (usually) to be solved numerically. The student would be provided with tools to do so. Being able to program in FORTRAN, or willing to learn, would be useful. Possible projects of current interest could be:

     a) Holographic superconductors with competing order parameters

     b) Black holes and compact objects in extended gravity models

  2. Black holes and other compact objects are accreting matter from their environment which leads to observable and quantifiable effects. The student would be investigating accretion processes around compact objects such as black holes and neutron stars by using large scale simulations and studying the observable effects. Good numerical skills would be very useful.

- **Dr David Hewett** d.hewett@ucl.ac.uk

  1. *Function spaces on fractal sets*

     The study of spaces of functions defined on subsets of Euclidean space R^n is a central topic in mathematical analysis. Standard examples include the spaces C^k of functions possessing a certain number of continuous classical derivatives, and the Sobolev spaces $W^{k,p}$ of functions possessing a certain number of weak derivatives whose pth powers are integrable. (These two examples are connected by the celebrated Sobolev embedding theorem, which says that $W^{k,p}$ is embedded in C^r provided $k > (n + r)/p$.) The study of function spaces becomes even more interesting (and challenging) when the domain on which the functions are defined is non-smooth (i.e. not a smooth sub-manifold of R^n). In this project the student would study functions defined on "fractal" sets possessing certain self-similarity properties, two well-known examples being the middle third Cantor set and the Sierpinski triangle. The student would learn about wavelet decompositions of function spaces on such sets, and study some recent results concerning the approximation properties of certain discrete (finite-dimensional) subspaces of them. An ambitious goal would be to construct and analyse new discrete spaces of higher order smoothness that would be of interest for the numerical solution of integral equations on fractal domains, a practical application of which is the study of light scattering by atmospheric ice crystals and snowflakes.

Analysis 4 (MATH0051) is an essential prerequisite. Ideally, the student should also have taken one or more of Linear PDEs (MATH0070), Measure Theory (MATH0017) and Functional Analysis (MATH0018).

- **Dr Duncan Hewitt** d.hewitt@ucl.ac.uk

  1. *Modelling pressure diffusion below ice sheets*

     Understanding the mechanics of glaciers and ice sheets is important for predicting the impact of climate change, and the conditions at their base play a key role in controlling the motion of the overlying ice. Transmission of pressure and transport of water through the porous sediment or 'till' below glaciers affects the drag on the base of ice sheets. This project will review and develop mathematical models to describe fluid motion through this sediment layer, and explore their implications in cases of varying till permeability or topography.

  2. *Mud mobilisation: transport of 'plastic' fluids in conduits*

     So-called 'non-Newtonian' fluids can exhibit complex and unusual behaviour. This project will explore the flow of visco-plastic fluids down channels or pipes; these materials, which include muds and clays as well as many household substances (shampoo, toothpaste, ketchup), can 'clog-up' and jam when the applied stress on them is too small. This property makes flows in channels very interesting, since either regions of the flow, or the whole flow, can jam up if the geometry or applied forcing changes. This project will study this behaviour theoretically, using a range of mathematical methods and asymptotic analysis.

  3. *Double-diffusive convection in porous media*

     The presence of two different sources of buoyancy that diffusive at different rates (e.g. heat and salt) can lead to interesting and complex convective behaviour in fluids. Such 'double-diffusive' convection is very important for ocean mixing, but it can also occur at much slower scales in underground flow through porous rocks, most notably in the vicinity of mid-ocean-ridges. In this project, the student will review the linear-stability problem of double-diffusion in a porous medium and then explore more complex behaviour when the driving forces are larger.

  All projects require knowledge from the methods courses, and all involve fluid mechanics, so some background knowledge of 2nd-year Fluid Mechanics (MATH0015) will be helpful. Many of the ideas in Real Fluids (MATH0077) will be relevant, so it might be sensible to take this course if you haven't yet.

- **Prof. Ted Johnson** e.johnson@ucl.ac.uk

  1. *Nonlinear wave equations*

     Solving nonlinear wave equations can present particular challenges due to the presence of rapidly propagating waves. A highly efficient method has been developed to numerically integrate these equation using exponential integrators. This project aims to summarise these methods and then apply them to some well known equations.

     Prerequisites: Ability to program in Matlab or Python.

2. *Time-periodic solutions of nonlinear differential equations*

   The problem of finding time-periodic solutions of nonlinear partial equations is non-trivial. This project uses an adjoint method to solve these problems by solving a series of initial value problems. This project is similar to project 1 but is harder.

   Prerequisites: Ability to program in Matlab or Python.

3. *Rossby-wave-linked basins* The dynamics of Rossby waves in a single ocean basin are well understood. This project aims to discuss how Rossby wave energy moves between two interlocking basins. The project will involve matching Fourier series to derive a matrix equation and then finding eigenvalues and eigenvectors using a standard Matlab or Python subroutine.

   Prerequisites: A knowledge of Geophysical Fluid dynamics MATH0024, and the ability to program in Matlab or Python.

4. *Modulational Instability*

   Provided the water is not too shallow a train of perfectly periodic water waves will eventually develop non-periodic features due to an instability labelled Modulational Instability (MI). This form of instability is often suggested as the cause of freak waves that can overwhelm ships. This project will review the literature on MI and then apply the theory to some well-known wave equations like the Korteweg-de Vries equation.

5. *Laplace's equation in domains with corners*

   This project considers a method for computing potential flows in planar domains put forward by Peter Baddoo. The approach is based on a new class of techniques, known as "lightning solvers", which exploit rational function approximation theory in order to achieve excellent convergence rates. The method is particularly suitable for flows in domains with corners where traditional numerical methods fail. The project will outline the mathematical basis for the method and establish the connection with potential flow theory. In particular, the new solver will be applied to a range of classical problems including steady potential flows, vortex dynamics, and free-streamline flows. The solution method is extremely rapid and usually takes just a fraction of a second to converge to a high degree of accuracy. Numerical evaluations of the solutions can be performed in a matter of microseconds and can be compressed further with novel algorithms. The method is described in the paper *'Lightning Solvers for Potential Flows', Peter J. Baddoo, Fluids 2020, 5, 227; doi:10.3390/fluids5040227*

   Prerequisites: Complex Analysis, Matlab or Python skills

- **Prof. Francis Johnson** f.johnson@ucl.ac.uk

  1. *Cohomology groups of finite groups*
  2. *Algebraic Topology*
  3. *Representation Theory*

- **Dr Ilia Kamotski** i.kamotski@ucl.ac.uk

  1. *Topics in homogenisation theory*

     Prerequisite: Linear Partial Differential Equations (MATH0070)

- **Dr Michal Kwasigroch** m.kwasigroch@ucl.ac.uk

  1. *Mathematical models of quantum spin liquids* A quantum spin liquid is an array of microscopic magnetic moments that remain strongly fluctuating, like a liquid, down to absolute zero temperature. Its state cannot therefore be easily parametrised in terms of the moment orientations. It turns out that the most natural description is in terms of 'fractions' of the magnetic moments. These natural degrees of freedom cannot be built from the microscopic constituents! The 'fractions' also carry a topological charge and interact with a background gauge field. The project will investigate and construct models for the dynamics of quantum spin liquids, which involves the propagation of the above topological defects in the background of gauge fields. Different analytical and numerical approaches can be taken in this project such as approximate or numerical solutions of many-body equations of motion and Monte Carlo methods. Knowledge of quantum and statistical mechanics would be advantageous but not essential.

     [1] C. Broholm et al., Science 367, 6475 (2020).

     [2] L. Savary and L. Balents, Rep. Prog. Phys. 80, 016502 (2017).

  2. *Mathematical models of quantum critical points*

     Fluctuations have a huge influence on the state of matter. They can even become infinitely strong if the external conditions are just right. Such conditions are known as quantum critical points and describe an amazing fractal state, where infinite fluctuations give birth to novel forms of matter. This is why quantum critical points are the matter analogue of stem cells. A particularly fruitful type of matter 'stem cell' is the ferromagnetic quantum critical point. The project will explore some of the possible mechanisms behind the formation of ferromagnetic quantum critical points. Different analytical and numerical approaches can be taken in this project such as self-consistent approximations, asymptotic methods and perturbation theory. Knowledge of quantum and statistical mechanics would be advantageous but not essential.

     [1] P. Coleman and A. J. Schofield, Nature 433, 226 (2005). [2] B. Shen et al., Nature 579, 51–55 (2020).

  3. *Mathematical models of quantum fluids*

     A quantum fluid is characterised by long-range quantum coherence or entanglement between its distant parts. Therefore, the fluid has to be described by a complex wavefunction rather than a velocity field. This gives rise to many exotic phenomena such as quantisation of vorticity in rotating superfluids. Percolation is the phenomenon by which a fluid moves through a network that has some of its links removed. There is sometimes a non-zero critical link density, knows as the percolation threshold, below which the fluid is completely unable to traverse the network. The project will explore percolation in the context of quantum liquids and try to address the question of whether a percolation threshold can exist for a quantum liquid but not for a classical one. Different analytical and numerical approaches can be taken in this project such as self-consistent approximations, variational ansatze, Monte Carlo methods or exact diagonalisation of large sparse matrices. Knowledge of quantum and statistical mechanics would be advantageous but not essential.

     [1] Thomas Vojta and Jörg Schmalian, Phys. Rev. Lett. 95, 237206 (2005). [2] C. Zhang and B. Capogrosso-Sansone, Phys. Rev. A 98, 013621 (2018).

  4. *Stochastic PDEs and thermalisation of quantum systems*

     The project will first review some of the generalisations of the Langevin equation, a stochastic PDE also used to model financial phenomena, to quantum systems. The project will then simulate the evolution of a quantum system in contact with an environment using one of these generalisations. Of particular current interest here, is the interplay between noise, friction and quantum entanglement, which the project will investigate. Entanglement presents many unique challenges to simulating quantum systems because of the exponential growth of the number of degrees of freedom with system size. A mixture of variational and Monte Carlo approaches will be used in an attempt to overcome or circumvent these challenges and simulate the stochastic equations. The project will also explore, time permitting, whether any lessons learned from the application of the Langevin equation to quantum systems can be transferred to financial systems.

[1] F. Reif, Fundamentals of Statistical and Thermal Physics, McGraw Hill New York (1965). (see section 15.5 Langevin Equation)

[2] K. Kanazawa et al., Phys. Rev. E 98, 052317 (2018).

[3] G. W. Ford and M. Kac, Journal of Statistical Physics 46, 803 (1987).

- **Dr Shoham Letzter** s.letzter@ucl.ac.uk

  1. A graph G is said to be universal for a family F of graphs if G contains every graph in F as a subgraph. Here the challenge is to construct a small universal graph (e.g. with as few edges as possible). This project will examine constructions, both classical and recent, of small universal graphs for various families of graphs.
  2. How many edges can a graph on n vertices without a copy of a fixed graph F have? This is a key question in extremal graph theory which is particularly challenging and interesting when F is bipartite. This project will focus on the above question in the case where F is complete bipartite and will possibly consider related variants.

  These projects are suitable for students who took Graph Theory and Combinatorics MATH0029.

- **Dr Lars Louder** l.louder@ucl.ac.uk

  1. *Stallings' theorem on groups with infinitely many ends and groups of cohomological dimension 1*
  2. *Diophantine problems in the free group, or 10,000 ways to write a commutator*

     The word w=uuvuvvUUVUVV in the free group ¡u,v¿ can be written as a commutator in two essentially distinct ways: w=[uuvuv,vUU]=[uuvu,vvU] (check it!). It turns out that any commutator can be written, up to some natural equivalence which comes from the topology of compact surfaces, in only finitely many ways, and up to a slightly coarser, but still natural, equivalence, in at most 10,000 ways.

     The aim of the project will be to use this as an introduction to studying equations over the free group. A more geometrically inclined student could push this project in the direction of algebraic geometry over groups, and a student who leans towards computer science and is already a capable coder could feasibly attempt to reduce the bound of 10,000, possibly all the way down to 2 (I hope!), using linear programming and branch and bound techniques a la the proof of the Kepler conjecture.

  Pre-requisites: Topology and Groups (MATH0074). Recommended: Geometry and Groups (MATH0052).

- **Dr Elena Luca** e.louca@ucl.ac.uk

  1. *Viscous flows in channel geometries*

     There has been much interest in understanding viscous flows in small scales motivated by microfluidic applications, low-Reynolds number swimming (e.g. dynamics of small organisms, such as bacteria), superhydrophobic surfaces etc. In two dimensions, one can rely on powerful complex analysis techniques to find solutions to the associated biharmonic equation $\nabla^4\psi(x,y) = 0$. The aim of this project is to study, using a mixture of analytical and numerical techniques, viscous flows in complicated channel geometries which have direct biological and medical applications. The computational aspects of the project will be performed using MATLAB.

     Pre-requisites: Fluid mechanics (MATH0015) and Complex analysis (MATH0013).

- **Dr Angelika Manhart** a.manhart@ucl.ac.uk

  1. *Pattern formation in collective dynamics*

     Pattern formation in collective dynamics of a large number of individuals, such as people, bacteria or sperm, is an ubiquitous phenomenon. In this project we will look at a system of obstacles and self-propelled swimmers. To analyse its properties, we will use both numerical tools and analytical methods (e.g. linear stability analysis).

     Prerequisites: Differential equations, Fourier transform, programming experience.

- **Dr Jonathan Marshall** jonathan.marshall@ucl.ac.uk

  1. *Investigating "secondary" Schottky groups*

     The theory of Schottky groups and automorphic functions, allied with conformal mapping, can be applied to construct explicit solutions to certain classes of problems set in planar multiply-connected domains, many of which are of physical interest. Recently, a special class of these groups - referred to as "secondary" Schottky groups – has been identified. The aim of this project is to investigate these groups and their possible uses in problems of applied mathematics.

     Prerequisites: Complex analysis (MATH0013), and familiarity with Matlab.

- **Prof. Robb McDonald** n.r.mcdonald@ucl.ac.uk

  1. *Interface growth in two dimensions*

     The nonlinear dynamics when an interface deforms in response to a quantity diffusing toward it generates remarkable patterns e.g. viscous fingering, branching stream networks and fractal-like structures formed in electro-deposition. This project will use complex analysis and simple numerical models to explore related models, such as Loewner growth, diffusion-limited aggregation, needle models, and the connections between them.

     Pre-requisites: Fluid mechanics (MATH0015) and Complex analysis (MATH0013). Willingness to use and adapt existing numerical models.

2. *Vortex dynamics*
   Investigate and develop analytical and numerical constructions of equilibria for the 2D Euler equations having non-zero vorticity distributions in the form of points, sheets and patches.
   Pre-requisites: Knowledge and enthusiasm for Fluid mechanics (MATH0015 and Complex analysis (MATH0013) is essential.

- **Dr Yusra Naqvi** y.naqvi@ucl.ac.uk

  1. *Positivity properties of symmetric polynomials*
     Symmetric polynomials play an important role in the representation theory of symmetric groups and Lie groups. In this project, we would explore a specific family of polynomials (such as Schubert polynomials or Macdonald polynomials) and explain observed positivity of their coefficients relative to a fixed basis by describing them through combinatorial means. This would build on popular models in combinatorics such as Young tableaux, crystals bases and folded alcove walks.
     Prerequisite: MATH0053

- **Prof. Karen Page** karen.page@ucl.ac.uk

  Projects offered in mathematical biology, in the general areas of evolutionary games, evolutionary dynamics and modelling embryonic development.
  Pre-requisites: Experience with differential equations, probability and stochastic processes and computer programming are all advantageous.

- **Prof. Leonid Parnovski** l.parnovski@ucl.ac.uk

  1. *Periodic operators and lattice points counting*
     Pre-requisites: Functional Analysis (MATH0018), Multivariable Analysis (MATH0019). Concurrent enrolment in Spectral Theory (MATH0071) would also be desirable.

- **Dr Philip Pearce** Philip_Pearce@hms.harvard.edu

  1. *Modelling blood flow in vascular networks*
     An intricate network of vessels transports blood between the heart and the rest of the organs in the human body. This project will begin with a review of theoretical models for blood flow in single idealised tubes and in networks of small blood vessels called capillaries. The aim will be to write code in e.g. Python or Matlab to simulate blood flow in various network topologies, and if possible to test how different assumptions about blood properties can be incorporated into such models.
     Pre-requisites: some programming experience; Real Fluids (a co-requisite).
  2. *Multi-scale modelling of diffusion*
     Understanding diffusion is vital in biological applications, from cell signalling to oxygen exchange in the lungs. This project will begin with a review of theoretical models for diffusion

via random motion. The aim will be to test how assumptions about random diffusive motion affect macroscopic properties of diffusive transport (for example, how bias in a random walker's direction of motion affects the partial differential equation governing the distribution of a cloud of walkers). This could involve either analytical methods, simulations in e.g. Python or Matlab, or a combination of the two methods.

Pre-requisites: Mathematical Methods 4

- **Prof. Yiannis Petridis** i.petridis@ucl.ac.uk

  1. *Lattice counting problems in Euclidean and hyperbolic spaces*
  2. *Ergodic theory of continued fractions*
  3. *The Erdos-Kac theorem on the number of distinct prime factors of the natural number n*
  4. *Selberg's theorem on the normal distribution of the Riemann-zeta function on its critical line*
  5. *L-functions of elliptic curves and effective bounds on class numbers of quadratic fields*

  Pre-requisites: Projects normally require Prime Numbers and their Distribution (MATH0083). Depending on the project, Elliptic Curves (MATH0036) or Geometry and Groups (MATH0052) may be useful. For 2. Functional Analysis (MATH0018) is useful.

- **Dr Ian Petrow** i.petrow@ucl.ac.uk

  1. *100% of Galois groups over Q are S_n.*

     When we study Galois theory, we learn to compute the Galois group of a polynomial, or more generally a finite extension of fields. The Galois group of a degree n irreducible polynomial is always a subgroup of the symmetric group S_n. It is natural to ask 'Which subgroups of S_n occur as Galois groups?', and if you just start to write down examples by picking a polynomial 'at random', you will find that you very often get the whole of S_n as its Galois group. In this project, you will make that idea precise and show that, if one orders polynomials of fixed degree with integer coefficients by the maximum absolute value of their coefficients, then, as size of the coefficients gets large, the proportion of polynomials with Galois group the full S_n approaches 100%. The proof of this fact is a beautiful mix of algebraic number theory, analysis (the large sieve), group theory, and prime number theory.

     Prerequisites: Required: Galois Theory (MATH0022) and Number Theory (MATH0022).

     Strongly recommended: Algebraic Number Theory (MATH0035) and Prime Numbers and their distribution (MATH0083).

  2. *Representation of integers by quadratic forms and modular forms*

     Let $Q(x)$ be a positive-definite quadratic form with integer coefficients in at least 3 variables. A basic and old question for each $n > 0$ is how many integral representations of n by the quadratic form $Q(x)$ are there? For some highly structured specific choices of Q, we can give an exact formula for the number of x in Z^r such that $n = Q(x)$, but in general an exact formula isn't possible. Instead, we look for approximate formulas for the number of solutions as $n \to \infty$. For quadratic forms in 3 or 4 variables the proof of such a formula uses modular forms, which are certain special functions on hyperbolic spaces which have deep connections to modern number theory. The goal of this project is to learn the proof of the asymptotic formula for the representation number and use this as motivation to learn the theory of modular forms. Of particular interest will be certain examples called theta functions, and their role in the proof of the representation number theorem.

Prerequisites: MATH0051 Analysis 4: Real Analysis MATH0052 Geometry and Groups MATH0083 Prime Numbers and their Distribution

- **Dr Alexey Pokrovskiy** a.pokrovskiy@ucl.ac.uk

1. *Cycles in sparse graphs*

   Many classic results in graph theory are about the existence of cycles in graphs with lots of edges (like Dirac's Theorem). This project will be about similar questions about graphs with relatively few edges (so called "sparse graphs").
2. *Sunflowers in hypergraphs*

   A sunflower is a collection of sets such that the intersection of any pair of them is exactly the same. There are several important problems in combinatorics along the lines of "show that every hypergraph with lots of edges contains a large sunflower". This project will be about looking at some problems like this (some of which have seen exciting breakthroughs recently).
3. Other problems about extremal combinatorics: other projects similar to the above are possible, for example on the topics of "twin-width of graphs" or "hat games on graphs".

These projects are all suitable to students who have taken Graph Theory and Combinatorics MATH0029.

- **Dr Ruth Reynolds** ruth.reynolds@ucl.ac.uk

1. *Topics in Noncommutative Algebra*

   The following are a suggestion of possible project directions, but I am happy to discuss projects more tailored to the student's specific interests:

   - Noncommutative fields of fractions and Goldie's Theorem;
   - The noetherianity of skew polynomial rings;
   - AS regular algebras and twisted homogeneous coordinate rings;
   - Weyl algebras.

   Suggested prerequisites: Algebra 4, Commutative algebra.

   References:

   Goodearl and Warfield, An introduction to noncommutative Noetherian rings.

   D. Rogalski, An Introduction to Noncommutative Projective Geometry.

   S.C. Coutinho, A Primer of Algebraic D-modules.

- **Dr Mark Roberts** m.l.roberts@ucl.ac.uk

  Projects in non-commutative ring theory. Possible topics include:

1. *Unique factorisation domains*

2. *Skew fields of fractions*

   Both generalise well-known commutative constructions to the non-commutative case.

   Pre-requisites: Algebra 4 (MATH0053) and at least two from Commutative Algebra (MATH0021), Galois Theory (MATH0022), Algebraic Topology (MATH0023), Algebraic Number Theory (MATH0035), Representation Theory (MATH0073), preferably including Commutative Algebra (MATH0021).

- **Dr Edward Segal** e.segal@ucl.ac.uk Topics in Geometry, Topology and Algebra:

  I'm happy to discuss potential projects in algebraic geometry, differential geometry, algebraic topology or algebra. A couple of examples are below.

  1. *Representations of quiver algebras*

     Pre-requisites: Commutative Algebra (MATH0021), Representation Theory (MATH0073).
  2. *Principal bundles*

     Pre-requisites: Differential Geometry (MATH0020). Helpful: Topology and Groups (MATH0074).

- **Dr Nadia Sidorova** n.sidorova@ucl.ac.uk

  1. *Reinforced random processes*
  2. *Branching processes*

  Prerequisites: Probability (MATH0069).

- **Prof. Michael Singer** michael.singer@ucl.ac.uk

  1. *Geometry of Classical and quantum mechanics*

     In this project we shall explore geometric quantization: symplectic geometry is the correct setting for classical mechanics. Geometric quantization is a recipe (though more of an art-form) for constructing the Hilbert spaces of quantum theory starting from a symplectic manifold.

     Pre-requisite: Multivariable calculus (MATH0019), desirable: Differential geometry (MATH0020). Useful: Analytical Dynamics (MATH0054).
  2. *Other projects in differential geometry*

- **Prof. Frank Smith** f.smith@ucl.ac.uk

  The project(s) will be chosen from the following three areas:

  1. *Industrial modelling problems such as in the internal and external flows of fluid associated with vehicle movements on land, sea or air*
  2. *Biomedical flows such as through branching vessels or flexibly walled vessels*

3. *Modelling related to sports such as for balls, bouncing and vehicle movements*

   Pre-requisites: the projects above are suitable for students who have taken a full range of methods courses, have experience with the theory of fluids and are interested in ap plying mathematics.

- **Prof. Valery Smyshlyaev** v.smyshlyaev@ucl.ac.uk

  1. *High frequency scattering: asymptotic methods and analysis*

     Problems of wave scattering are mathematically boundary value problems for a PDE. Their approximate solutions for high frequencies can be constructed analytically by a multivariable version of WKB method, which is one of asymptotic methods. Such approximations have a clear physical meaning, and tools of analysis are needed for controlling the accuracy of these approximations.

     Pre-requisites: Waves and Wave Scattering (MATH0080) and Analysis 4 (MATH0051);

  2. *Multi-scale problems and homogenisation: asymptotic methods and analysis*

     Nearly everything around us contains multiple scales, i.e. has often invisible microscopically varying physical properties on which their visible macroscopic properties depend. Mathematically, one needs to deal with boundary-value problems for PDEs with microscopically varying coefficients, and then homogenisation becomes the process of deriving approximate PDEs with macroscopic coefficients. One way of doing this is via asymptotic methods with respect to the underlying small parameter, and the resulting approximations often display interesting physical effects. Tools of analysis are needed for controlling the accuracy of such approximations.

     Pre-requisites: Functional Analysis (MATH0018) and Mathematical Methods 4 (MATH0056).

- **Prof. Alex Sobolev** a.sobolev@ucl.ac.uk

  1. *Pseudo-differential operators*
  2. *Mathematical theory of wavelets*

     Prerequisites: Analysis 4: Real Analysis (MATH0051), Functional Analysis (MATH0018).

- **Prof. Alan Sokal** a.sokal@ucl.ac.uk

  1. *Total positivity in enumerative combinatorics*

     A matrix of real numbers is called totally positive if all its minors are nonnegative. A matrix of polynomials (with real coefficients) is called coefficientwise totally positive if all its minors are polynomials with nonnegative coefficients. It turns out that many matrices (of numbers or polynomials) arising in enumerative combinatorics are (or seem to be) totally positive: sometimes we know how to prove this, but in many other cases it is merely a conjecture supported by numerical tests.

     In this project the student will learn some of the basic theory of total positivity and its applications to enumerative combinatorics, and then do some numerical explorations with the aim of discovering new examples of total positivity (and hopefully proving them!).

2. *Continued fractions for formal power series and their applications to enumerative combinatorics*

   In 1746, Euler showed how the series $\sum_{n=0}^{\infty} n!\, t^n$ can be expanded as a continued fraction. (This series has zero radius of convergence, but who cares? We treat it as a formal power series: a concept that has a perfectly rigorous meaning in abstract algebra.) In the two-and-a-half centuries since then, many continued fractions have been found for well-known special functions and for generating functions arising in combinatorial enumeration.

   In this project the student will learn some of the basic theory of continued fractions for formal power series, and their applications to enumerative combinatorics, including algebraic and combinatorial methods of proof. Ideally the student will also do some numerical explorations with the aim of discovering new examples of continued fractions (and hopefully proving them!).

Both of these projects are highly interdisciplinary: they require a firm grounding in elementary aspects of linear and abstract algebra, real and complex analysis, and graph theory. Above all, you will need to acquire quickly a basic familiarity with enumerative combinatorics (enumeration of permutations, set partitions, trees and forests, etc.), at the level of Graham–Knuth–Patashnik, *Concrete Mathematics*, if this material is not already familiar to you. A good knowledge of Mathematica (MATH0032) will be essential for the numerical explorations.

- **Dr David Solomon** d.r.solomon@ucl.ac.uk

1. *p-adic Numbers, p-adic Analysis and Applications to Number Theory*

   This project lies intriguingly at the interface of Analysis and Number Theory. The 'p' in the title is a prime number and to each such p there corresponds a topologically complete field, the p-adic numbers. It is a bit like the field of real numbers but with some very striking – and, initially, rather disorienting - differences. In particular, it is 'non-Archimedean'. In this project, we will first construct the p-adic numbers then do some elementary p-adic analysis with them including the construction of p-adic logarithmic, exponential and Gamma functions. These turn out to be far easier to handle than the real and complex functions of which they are analogues, which is fortunate when we apply the Analysis back to the study of Number Theory. One application uses the p-adic Gamma-function to prove 'supercongruences' with respect to p for binomial coefficients. Another, more ambitious, uses p-adic L-functions to study the p-divisibility of the class-numbers of certain number fields.

   Pre-requisites: MATH0034. Also, MATH0035, depending on applications chosen. Plus a solid understanding of standard 1st and 2nd-year Real and Complex Analysis.

2. *Binary Quadratic Forms, Prime Numbers and Class-Groups*

   An integral binary quadratic form (IBQF) is a homogeneous quadratic expression in two variables with whole-number coefficients. Today, we view 'class-groups', 'units' etc. as lying in the domain of algebraic number fields but historically speaking, their origins lie in the study of IBQFs by the great Carl Friedrich Gauss (1777-1855) as well as other 18th and 19th-century mathematicians. Today, IBQFs and their generalisations remain relevant to both computational and theoretical number theory. This project will start by examining the basic notions of IBQFs: definiteness, representability and equivalence. We will look at the very classical question of which prime numbers are values of such forms (generalising the famous two-squares theorem) which leads us to a first glimpse of class-field theory. We will also study the application of IBQFs to the computation of class-groups for quadratic number fields. Pre-requisites: MATH0034 and MATH0035.

3. *Cyclotomic and Abelian Fields with Applications*
   A cyclotomic field is a number field generated over the rationals by a root of unity. Such fields form one of the most intensively studied classes of number fields because of their explicit nature and the fact that any abelian field - a Galois extension of the rationals with abelian Galois group - is contained in a cyclotomic field. (This is the Kronecker-Weber Theorem.) First, we will analyse in more detail the objects from MATH0035 in the specific case of cyclotomic fields: rings of integers, class groups and units but also objects peculiar to cyclotomic fields: Jacobi Sums, Gauss Sums, and Cyclotomic units, for instance. Emphasis will be placed on their structure as modules for the Galois group. We will then apply the theory to one or more of the following: Stickelberger's Theorem, Thaine's Theorem, proof of the Catalan Conjecture, certain pre-Wiles cases of FLT and the Kronecker-Weber Theorem.
   Pre-requisites: MATH0034, MATH0035 and MATH0053. Some ideas from MATH0073 would also useful.

- **Dr Isidoros Strouthos** i.strouthos@ucl.ac.uk

  1. Topics in abstract algebra (related, for instance, to algebraic K-theory and/or the theory of group rings)
     Even though different relevant projects might correspond to different (collections of) relevant pre-requisite modules, such projects are, in general, due to involve material covered in modules such as 'Algebra 3: Further Linear Algebra' and 'Algebra 4: Groups and Rings', as well as material covered in at least one of the modules 'Commutative Algebra', 'Representation Theory'; please feel free to contact the potential supervisor directly for more information regarding possible relevant projects.

- **Dr John Talbot** j.talbot@ucl.ac.uk

  1. *Counting cliques in graphs*
     How many triangles must a graph of given order and size contain? Recently [3] a very general version of this question was answered. This highly technical result was the culmination of many decades of progress on this topic starting with Rademacher in 1941. A project in this area could take many different directions, including looking at special cases and related algorithmic questions.
     [1] Nikiforov, Vladimir. The number of cliques in graphs of given order and size. Trans Am Math Soc. (2007) 363. 1599-1618
     [2] Razborov, Alexander. On the Minimal Density of Triangles in Graphs. Combinatorics, Probability and Computing, 17(4), (2008), 603-618
     [3] Reiher, Christian. The Clique Density Theorem. Annals of Mathematics, vol. 184, no. 3, (2016), pp. 683–707. Second Series.
     Prerequisites: MATH0029 (Graph Theory and Combinatorics).

- **Dr Sergei Timoshin** s.timoshin@ucl.ac.uk

1. *Two-fluid flows*

   Two-fluid flows can be studied in various approximations which reflect the specifics of the flow (e.g. thin layers), in two and three dimensions, with or without explicit time dependence. There are many interesting and unsolved problems related, for example, to flow separation and instability.

   Prerequisites: Knowledge of fluid dynamics at the level of Real Fluids (MATH0077) is essential.

- **Dr Matthew Towers** m.towers@ucl.ac.uk

  1. *Universal enveloping algebras*

     There is a ring U, called the 'universal enveloping algebra', associated to each Lie algebra L. The project will study the universal enveloping algebras associated to the special linear Lie algebra sln for $n = 2$ or 3 over fields of characteristic 2. The specifics will depend on the interests and the background of the student undertaking the project: for example, we might look at the centres of these algebras, their algebras of derivations, or their cohomology.

     Pre-requisites: Algebra 4 (MATH0053), and some of Commutative Algebra (MATH0021), Algebraic Topology (MATH0023), Representation Theory (MATH0073). Recommended concurrently: Lie groups and Lie algebras (MATH0075).

  2. *Category theory and topology in functional programming*

     One possible project begins by giving an introduction to category theory in the context of functional programming. After that it would investigate semantic approximation order, recursive definitions as fixed points, or monads and their algebras.

     Another idea would be to give an account of some remarkable work of Martin Escardo https://www.cs.bham.ac.uk/~mhe/.talks/popl2012/escardo-popl2012.pdf on decidable equality for function types.

     Pre-requisites: some programming skills, and knowledge of or willingness to learn a functional language, e.g. Haskell, Scheme.

- **Prof. Jean-Marc Vanden-Broeck** j.vanden-broeck@ucl.ac.uk

  1. *Analytical and numerical studies of waves of large amplitude*

     Pre-requisites: Fluid Mechanics (MATH0015) or equivalent.

- **Prof. Alexey Zaikin** alexey.zaikin@ucl.ac.uk

  1. *Modelling stochastic neuron-astrocyte networks*

     Recent investigations discuss that the human brain may use artificial intelligence algorithms in information processing functioning [1]. This is based on artificial intelligence principles working in spiking neural networks [2; 3]. Additionally, we know that human brain is a network of networks containing overlapping networks of neurons and astrocytes [4]. Astrocyte locally mediate coupling between neurons by affecting neurotransmitters through calcium events. This process can be modelled by well-established approaches [4]. Also, it was shown that astrocytes can induce auto-associative memory in the neuron-astrocyte system [5]. On the other hand, some time ago we have shown that excitable regime can be induced by joint action of noise and coupling [6]. The aim of the project is to investigate noise-induced excitability in the presence of astrocytes and find which AI functions can be achieved in such a system.

[1] J. Guerguiev, T.P. Lillicrap, and B.A. Richards, Towards deep learning with segregated dendrites. Elife 6 (2017).

[2] A. Samadi, T.P. Lillicrap, and D.B. Tweed, Deep learning with dynamic spiking neurons and fixed feedback weights. J Neural computation 29 (2017) 578-602.

[3] W. Maass, T. Natschläger, and H. Markram, Real-time computing without stable states: A new framework for neural computation based on perturbations. J Neural computation 14 (2002) 2531-2560.

[4] O. Kanakov, S. Gordleeva, A. Ermolaeva, S. Jalan, and A. Zaikin, Astrocyte-induced positive integrated information in neuron-astrocyte ensembles. Physical Review E 99 (2019) 012418.

[5] S. Gordleeva, Y. Lotareva, M. Krivonosov, A. Zaikin, M. Ivanchenko, and A. Gorban, "Astrocyte organize associative memory", in "Advances in Neural Computation, Machine Learning, and Cognitive Research III", Springer Nature p. 384 (2019).

[6] E. Ullner, A. Zaikin, J. Garcia-Ojalvo, and J. Kurths, "Noise-Induced Excitability in Oscillatory Media", Phys. Rev. Lett. 91, 180601 (2003).

Pre-requisites: Biomathematics (MATH0026) and some programming skills. The project is basically computational and includes numerical simulations of Stochastic Differential Equations.